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EXTENSIONS IN INVEXITY THEORY

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ABSTRACT. In this paper we extend two properties of invex functions. On one hand, we establish that *a function is pseudoinvex if and only if every stationary point is a global minimum.* On the other hand, we give conditions for the invexity of real functions on nonempty arbitrary sets. The results are established in differentiable and nonsmooth variants, also.

1. INTRODUCTION

In 1977, Zang, Choo and Avriel [10] studied nonconvex functions whose stationary points are global and they applied them in mathematical programming. In 1981, Hanson [4] considered a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ for which there exists a vector function $\eta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\forall x, u \in \mathbb{R}^n : \quad f(x) - f(u) \geq \eta^t(x,u)\nabla f(u) \tag{1.1}$$

(^{t} denotes the transposition sign). It is clear that the function f which satisfies (1.1) generalizes the differentiable convex functions. Hanson [4] showed that weak duality and sufficiency of the Kuhn-Tucker conditions still hold if instead of the convexity conditions, the objective function and all constraints of a nonlinear programming problem satisfy (1.1) with the same η. In 1981, Craven [2] called the functions satisfying (1.1) *invex*, too.

2. FIRST EXTENSION: STATIONARY POINTS PROPERTY OF PSEUDOINVEX FUNCTIONS

In 1985, Craven and Glover [3] showed that the following theorem holds true (in the equivalent form given by Ben-Israel and Mond [1]):

Theorem 2.1. *The function f is invex if and only if every stationary point is a global minimum.*

Because of this important property, the invex functions have replaced the convex functions in optimization theory.

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Proof. The *necessity* is obvious.

Sufficiency. If $\nabla f(u) = 0 \Rightarrow f(x) \geq f(u)$, $\forall x \in \mathbb{R}^n$, then, according to Ben-Israel and Mond, f is invex with respect to the vector function $\eta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$\eta(x,u) = \begin{cases} 0 & \text{if} \quad \nabla f(u) = 0 \\ \dfrac{[f(x)-f(u)]\nabla f(u)}{[\nabla f(u)]^t \nabla f(u)} + v & \text{if} \quad \nabla f(u) \neq 0, \end{cases}$$

where $v^t \nabla f(u) \leq 0$. Indeed, we have

$$f(x) - f(u) - \eta^t(x,u)\nabla f(u) = -v\nabla f(u) \geq 0.$$

□

Now, we recall

Definition 2.1 (Hanson [4])**.** The differentiable function f is said to be *pseudoinvex* if there exists a vector function η such that

$$\forall x, u \in \mathbb{R}^n: \quad \eta^t(x,u)\nabla f(u) \geq 0 \Rightarrow f(x) \geq f(u). \tag{2.1}$$

An extension of Theorem 2.1 is the following:

Theorem 2.2. *The differentiable function f is pseudoinvex if and only if every stationary point is a global minimum.*

Proof. Necessity. Suppose that f satisfies (2.1) and $u \in \mathbb{R}^n$ is a stationary point of f, that is $\nabla f(u) = 0$. Then, $\forall x \in \mathbb{R}^n$, we have $\eta^t(x,u)\nabla f(u) = 0$ and according to (2.1) it results $f(x) \geq f(u)$, $\forall x \in \mathbb{R}^n$.

Sufficiency. The function f satisfies: $\nabla f(u) = 0 \Rightarrow f(x) \geq f(u)$, $\forall x \in \mathbb{R}^n$. We define $\eta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$x, u \in \mathbb{R}^n : \eta(x,u) = \begin{cases} 0 & \text{if} \quad \nabla f(u) = 0 & \text{(i)} \\ \dfrac{2[f(x)-f(u)]}{[\nabla f(u)]^t \nabla f(u)} \nabla f(u) & \text{if} \quad \nabla f(u) \neq 0. & \text{(ii)} \end{cases}$$

Then

(i) $\nabla f(u) = 0 \Rightarrow f(x) \geq f(u)$ is equivalent to

$$\eta^t(x,u)\nabla f(u) = 0 \Rightarrow f(x) \geq f(u).$$

(ii) $\nabla f(u) \neq 0$. Then

$$\eta^t(x,u)\nabla f(u) \geq 0 \Rightarrow \frac{2[f(x)-f(u)]}{[\nabla f(u)]^t \nabla f(u)} [\nabla f(u)]^t \nabla f(u) \geq 0 \Rightarrow$$

$$\Rightarrow f(x) - f(u) \geq 0 \Rightarrow f(x) \geq f(u).$$

Therefore, from (i) and (ii) it results (2.1), that is f pseudoinvex with respect to η. □

Remark 2.1. The function f is not invex with respect to η because of the equality

$$\eta^t(x,u)\nabla f(u) = 2[f(x) - f(u)].$$

Corollary 2.1. *If f has no stationary points, then f is pseudoinvex.*

Remark 2.2. Other expression of η for the pseudoinvexity of f:

$$\eta(x,u) = \begin{cases} 0 & \text{if} \quad \nabla f(u) = 0 \\ \dfrac{[f(x) - f(u)]\varphi(x,u)}{[\nabla f(u)]^t \nabla f(u)} \nabla f(u) & \text{if} \quad \nabla f(u) \neq 0, \end{cases}$$

where $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to (0,1) \cup (1,\infty)$.

3. SECOND EXTENSION: INVEXITY CONDITIONS OF REAL FUNCTIONS ON ARBITRARY SETS

According [9], we remark that the domain of a invex function is an open set. Let X be a nonempty arbitrary set in $\mathbb{R}^n$ and let $f\colon X \to \mathbb{R}$. Comparing the invexity of f on X, with the invex functions on open sets, the difference appears on the boundary of X and it concerns η, according to the following considerations. We denote the boundary of X by $\mathrm{Fr}\, X$, and the relative interior of X by $\mathrm{ri}\, X$.

Definition 3.1. A vector function $\tilde{\eta}\colon X \times X \to \mathbb{R}^n$ is called *a continuous extension of the vector function $\eta\colon ri\, X \times ri\, X \to \mathbb{R}^n$ on $X \times X$ if $\tilde{\eta}(x,u) = \eta(x,u)$,* $\forall (x,u) \in \mathrm{ri}\, X \times \mathrm{ri}\, X$ and for each $y, b \in (\mathrm{Fr}\, X) \cap X$ the limit $\lim\limits_{\substack{x,u \in \mathrm{ri}\, X \\ (x,u)\to(y,b)}} \eta(x,u)$ exists and it is equal to $\tilde{\eta}(y,b)$.

The next result holds true:

Theorem 3.1. *Let D be a nonempty open set in $\mathbb{R}^n$, $X \subset D$ a nonempty subset and $f\colon D \to \mathbb{R}$. We suppose that f is continuously differentiable on D and invex on* $\mathrm{ri}\, X$ *with respect to the vector function $\eta\colon \mathrm{ri}\, X \times \mathrm{ri}\, X \to \mathbb{R}^n$. If $\tilde{\eta}\colon X \times X \to \mathbb{R}^n$ is a continuous extension on $X \times X$ of η, then f is invex on X with respect to $\tilde{\eta}$.*

Proof. Since the function f is invex with respect to η on $\mathrm{ri}\, X$, we have

$$\forall x, u \in \mathrm{ri}\, X : \quad f(x) - f(u) \geq \eta^t(x,u)\nabla f(u). \tag{3.1}$$

Let $y, b \in (\mathrm{Fr}\, X) \cap X$. Taking all the variants $(x,u) \to (y,b)$ in (3.1) it results

$$\forall y, b \in (\mathrm{Fr}\, X) \cap X : \quad f(y) - f(b) \geq \tilde{\eta}^t(y,b)\nabla f(b). \tag{3.2}$$

From (3.1) and (3.2) it results

$$\forall x, u \in X : \quad f(x) - f(u) \geq \tilde{\eta}^t(x,u)\nabla f(u).$$

Therefore, f is invex on X with respect to $\tilde{\eta}$. $\square$

Theorem 3.1 can be generalized for pseudoinvex and quasiinvex functions.

Definition 3.2. The differentiable function $f\colon D \to \mathbb{R}$ is called *quasiinvex* on D if there exists a vector function $\eta : D \times D \to \mathbb{R}^n$ such that

$$\forall x, u \in D : \quad f(x) \leq f(u) \Rightarrow \eta^t(x,u)\nabla f(u) \leq 0.$$

For real functions, the following implications hold true [8]:

$$\text{invex} \Rightarrow \text{pseudoinvex} \Rightarrow \text{quasiinvex}.$$

Denote by $\operatorname{ri} X$ the relative interior of X. The following results hold true:

Theorem 3.2. *Let D be a nonempty open set in $\mathbb{R}^n$, $X \subset D$ a nonempty subset and $f\colon D \to \mathbb{R}$. We suppose that f is continuously differentiable on D and pseudoinvex on $\operatorname{ri} X$ with respect to vector function $\eta\colon \operatorname{ri} X \times \operatorname{ri} X \to \mathbb{R}^n$. If $\tilde{\eta}\colon X \times X \to \mathbb{R}^n$ is a continuous extension on $X \times X$ of η, then f is pseudoinvex on X with respect to $\tilde{\eta}$.*

Theorem 3.3. *Let D be a nonempty open set in $\mathbb{R}^n$, $X \subset D$ a nonempty subset and $f\colon D \to \mathbb{R}$. We suppose that f is continuously differentiable on D and quasiinvex on $\operatorname{ri} X$ with respect to vector function $\eta\colon \operatorname{ri} X \times \operatorname{ri} X \to \mathbb{R}^n$. If $\tilde{\eta}\colon X \times X \to \mathbb{R}^n$ is a continuous extension on $X \times X$ of η, then f is quasiinvex on X with respect to $\tilde{\eta}$.*

4. STATIONARY POINTS PROPERTY FOR NONSMOOTH PSEUDOINVEX FUNCTIONS

In this section we give the variant of Theorem 2.2 for nonsmooth functions.

Let A be a nonempty open set in $\mathbb{R}^n$ and the nonsmooth function $F\colon A \to \mathbb{R}$. The Dini upper directional derivative of f at $u \in A$ on direction $d \in \mathbb{R}^n$ is defined by

$$f'_+(u;d) = \limsup_{\lambda \searrow 0} \frac{f(u+\lambda d) - f(u)}{\lambda}.$$

Definition 4.1 (Komlósi [5])**.** A point $x^0 \in A$ is said to be a *stationary point* of f if $f'_+(x^0; v) \geq 0$, $\forall v \in \mathbb{R}^n$.

Definition 4.2 ([7])**.** The function f is said to be *pseudoinvex on A* if there exists a vector function $\eta : A \times A \to \mathbb{R}^n$ such that

$$\forall x, u \in A : \quad f'_+(x; \eta(x,u)) \geq 0 \Rightarrow f(x) \geq f(u). \tag{4.1}$$

If u is fixed, then f is called *pseudoinvex at u*.

It is known that the domain of a pseudoinvex function is an invex set [9], that is there exists $\eta \neq 0$ such that $\forall x, u \in A$, $\forall \lambda \in [0,1] \Rightarrow u + \lambda\eta(s,u) \in A$.

It follows the variant of Theorem 2.2 for nonsmooth functions.

Theorem 4.1. *The nonsmooth function f is pseudoinvex if and only if every stationary point of f is a global minimum point.*

Proof. Let A be an open invex set with respect to a vector function η.

Necessity. Suppose that f is pseudoinvex on A with respect to η and $u \in A$ is a stationary point of f. Then $f'_+(u; v) \geq 0$, $\forall v \in \mathbb{R}^n$. Particularly, for $v = \eta(x, u)$, where $x \in A$, it results

$$f'_+(u; \eta(x, u)) \geq 0, \quad \forall x \in A. \tag{4.2}$$

Taking into account (4.1), relation (4.2) implies $f(x) \geq f(u)$, $\forall x \in A$.

Sufficiency. There are two cases.

(i) Suppose that u is a stationary point of f. From our hypothesis we have

$$f'_+(u; v) \geq 0, \quad \forall v \in \mathbb{R}^n \Rightarrow f(x) \geq f(u). \tag{4.3}$$

Particularly, for $v = \eta(x, u)$, $\forall x \in A$, the relation (4.3) becomes

$$\forall x \in A: \quad f'_+(u; \eta(x, u) \geq 0 \Rightarrow f(x) \geq f(u).$$

Therefore, f is pseudoinvex at u.

(ii) u is not a stationary point. Then there exists a vector $\bar{v} \in \mathbb{R}^n$ such that $f'_+(u; \bar{v}) < 0$. In addition, we suppose that η is surjective. Then there exists an $\bar{x} \in A$ such that $\bar{v} = \eta(\bar{x}, u)$ and we have $f'_+(u; \eta(\bar{x}, u)) < 0$. In the definition of pseudoinvexity of f at u, this inequality is determined by the condition $f(\bar{x}) < f(u)$, that is

$$f(\bar{x}) < f(u) \Rightarrow f'_+(u; \eta(\bar{x}, u)) < 0. \tag{4.4}$$

We suppose the contrary, that is the function f is not pseudoinvex at u. Then, $f(\bar{x}) \geq f(u)$. Consequently,

$$f'_+(u; \eta(\bar{x}, u)) < 0 \Rightarrow f(\bar{x}) \geq f(u). \tag{4.5}$$

Since the relations (4.4) and (4.5) are contradictory, it conclude that the assumption made above, is false. It results that f is pseudoinvex at u. □

5. NONSMOOTH INVEX FUNCTIONS ON ARBITRARY SETS

Let $D \subseteq \mathbb{R}^n$ be an open set, $x \in D$ and $v \in \mathbb{R}^n$.

Proposition 5.1. *Suppose that f is continuous on a neighborhood of x and f'_+ is bounded on a neighborhood of (x, v). Then the function f'_+ is upper semicontinuous at (x, v).*

Proof. Let us consider the sequences $(x_k) \in D$, $x_k \to x$ and $(v_k) \subset \mathbb{R}^n$, $v_k \to v$. f'_+ being an upper limit at (x, v) then, for each k, there exists $\lambda_k > 0$ such that $||x_k - x|| + \lambda_k < \dfrac{1}{k}$ implies

$$f'_+(x_k; v_k) - \frac{1}{k} < \frac{f(x_k + \lambda_k v_k) - f(x_k)}{\lambda_k} =$$

$$= \frac{f(x_k + \lambda_k v) - f(x_k)}{\lambda_k} + \frac{f(x_k + \lambda_k v_k) - f(x_k + \lambda_k v)}{\lambda_k}. \tag{5.1}$$

We consider the sequence of continuous functions (φ_k), where $\varphi_k(v) = f(x_k + \lambda_k v)$. There exists a sequence $\delta\left(\lambda_s \frac{1}{s}\right) \searrow 0$ such that

$$\|v_s - v\| < \delta\left(\lambda_s \frac{1}{s}\right) \Rightarrow |\varphi_k(v_s) - \varphi_k(v)| < \lambda_s \frac{1}{s}.$$

For $s = k$, we obtain $\left|\frac{\varphi_k(v_k) - \varphi(v)}{\lambda_k}\right| < \frac{1}{k}$, or

$$\left|\frac{f(x_k + \lambda_k v_k) - f(x_k + \lambda_k v)}{\lambda_k}\right| < \frac{1}{k},$$

from which it results

$$\lim_{k\to\infty} \frac{f(x_k + \lambda_k v_k) - f(x_k + \lambda_k v)}{\lambda_k} = 0.$$

Taking this limit into account, from (5.1) we obtain

$$\begin{aligned}\limsup_{k\to\infty} f'_+(x_k; v_k) &\leq \limsup_{k\to\infty} \frac{f(x_k + \lambda_k v) - f(x_k)}{\lambda_k} \\ &\leq \limsup_{(y,\lambda)\to(x,0_+)} \frac{f(y + \lambda v) - f(y)}{\lambda}.\end{aligned}$$

Using the continuity of f, we have

$$\limsup_{k\to\infty} f'_+(x_k; v_k) \leq \lim_{\lambda\searrow 0} \frac{f(x + \lambda v) - f(x)}{\lambda} = f'(x; v) = f'_+(x, v).$$

Therefore, f'_+ is upper semicontinuous at (x, v). □

Definition 5.1 ([7])**.** The function f is said to be *invex on A* if there exists a vector function $\eta\colon A \times A \to \mathbb{R}^n$ such that

$$\forall x, u \in A: \quad f(x) - f(u) \geq f'_+(x; \eta(x, u)). \tag{5.2}$$

If u is fixed, then f is called *invex at u*.

The next result holds

Theorem 5.1. *Let D be an open set in $\mathbb{R}^n$, $X \subset D$ a nonempty subset and the function $f\colon D \to \mathbb{R}$ and $\eta\colon \mathrm{ri}\, X \times \mathrm{ri}\, X \to \mathbb{R}^n$. We suppose that the following conditions are fulfilled:*

1) *f is continuous on X;*
2) *$\tilde{\eta}\colon X \times X \to \mathbb{R}^n$ is a continuous extension on X of η;*
3) *f is invex on $\mathrm{ri}\, X$ with respect to η.*

Then f is invex on X with respect to $\tilde{\eta}$.

Proof. Since the function f is invex with respect to η on $\mathrm{ri}\, X$ we have

$$\forall x, u \in \mathrm{ri}\, X : \quad f(x) - f(u) \geq f'_+(u; \eta(x, u)). \tag{5.3}$$

Let $y, b \in \mathrm{Fr}\, X \cap X$. According to Proposition 5.1, the function f'_+ is upper semicontinuous at (x, u). We take the upper limit at (5.3) from $(x, u) \to (y, b)$ and it results

$$f(y) - f(b) \geq \limsup_{(x,u)\to(y,b)} f'_+(u; \eta(x, u)) = f'_+\left(b; \limsup_{(x,u)\to(y,b)} \eta(x, u)\right). \tag{5.4}$$

But $\tilde{\eta}$ is a continuous extension on X of η, then from (5.4) we obtain

$$f(y) - f(b) \geq f'_+(b; \tilde{\eta}(y, b)).$$

This inequality holds for any $(y, b) \in X \times X$. Therefore, f is invex on X with respect to $\tilde{\eta}$. $\square$

Theorem 5.2. *Let D be an open set in $\mathbb{R}^n$, $X \subset D$ a nonempty subset and the functions $f \colon D \to \mathbb{R}$ and $\eta \colon \mathrm{ri}\, X \times \mathrm{ri}\, X \to \mathbb{R}^n$. We suppose that the following conditions are fulfilled:*

4) *f is continuous on X;*
5) *$\tilde{\eta} \colon X \times X \to \mathbb{R}^n$ is a continuous extension on X of η;*
6) *f is pseudoinvex on* $\mathrm{ri}\, X$ *with respect to η.*

Then f is pseudoinvex on X with respect to $\tilde{\eta}$.

Definition 5.2 ([7])**.** The function f is said to be *quasiinvex on A* if there exists a vector function $\eta : A \times A \to \mathbb{R}^n$ such that

$$\forall x, u \in A : \quad f(x) \leq f(u) \Rightarrow f'_+(x; \eta(x, u)) \geq 0. \tag{5.5}$$

If u is fixed, then f is called *quasiinvex at u*.

Theorem 5.3. *Let D be an open set in $\mathbb{R}^n$, $X \subset D$ a nonempty subset and the functions $f : D \to \mathbb{R}$ and $\eta :$ $\mathrm{ri}\, X \times \mathrm{ri}\, X \to \mathbb{R}^n$. We suppose that the following conditions are fulfilled:*

7) *the function f is continuous on X;*
8) *$\tilde{\eta} \colon X \times X \to \mathbb{R}^n$ is a continuous extension on X of η;*
9) *f is quasiinvex on* $\mathrm{ri}\, X$ *with respect to η.*

Then f is quasiinvex on X with respect to $\tilde{\eta}$.

REFERENCES

[1] A. Ben-Isreal and B. Mond: *What is invexity?*, J. Austral. Math. Soc., Ser. B, **28**(1986), No. 1, 1-9.

[2] B. D. Craven: *Invex function and constrained local minima*, Bull. Austral. Math. Soc., **24**(1981), 357-366.

[3] B. D. Craven and B. M. Glover: *Invex function and duality*, J. Austral. Math. Soc., Ser. A, **39**(1985), 1-20.

[4] M. A. Hanson: *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl., **80**(1981), 545-450.

[5] S. Komlósi: *Some properties of nondifferentiable pseudoconvex functions*, Math. Programming, **26**(1983), 232-237.

[6] Şt. Mititelu: *Invex sets*, Stud. Cerc. Mat., **46**(1994), No. 5, 529-532.

[7] Şt. Mititelu: *Invex functions*, Rev. Roumaine Math. Pures Appl., **49**(2004), No. 5-6, 529-544.

[8] Şt. Mititelu: *Nonsmooth invex functions. The preinvex case*, Math. Reports, **8**(58)(2006), No. 4, 435-452.

[9] Şt. Mititelu: *Invex sets and nonsmooth invex functions*, Rev. Roumaine Math. Pures Appl., **52**(2007), No. 6, 665-672.

[10] I. Zang, E. U. Choo and M. Avriel: *On functions whose stationary points are global*, J. Optim. Theory Appl., **22**(1977), 195-208.

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