J. Adv. Math. Studies
Vol. **1**(2008), No. 1-2, 47-62
`http://journal.fairpartners.ro`

NONEXISTENCE OF THE CONFORMALLY FLAT PROJECTIVISED TANGENT BUNDLE OF A FINSLER SPACE WITH THE CHERN-RUND CONNECTION

HIROSHI ENDO AND SHIGEO FUEKI

Abstract. In a Riemannian manifold (M, g) K. Bang (e.g., [5] p. 141) showed that the tangent bundle TM with the Sasaki metric $\overline{g}$ is conformally flat if and only if the base manifold (M, g) is flat, in which case $(TM, \overline{g})$ is flat. For this fact, we consider the conformally flat projectivised tangent bundle PTM with the Sasaki metric on a Finsler space (M, F) with the Chern-Rund connection. Then it is shown that there does not exist a conformally flat projectivised tangent bundle PTM on a Finsler Space (M, F).

1. PRELIMINARY

Let M be an m-dimensional C^∞ manifold and x^i $(1 \leq i \leq m)$ be local coordinates on M. It is said to be a Finsler manifold if the length s of any curve $t \mapsto (x^1(t), \dots, x^m(t))$ $(a \leq t \leq b)$ is given by an integral

$$s = \int_a^b F\left(x^1(t), \dots, x^m(t), \frac{dx^1}{dt}, \dots, \frac{dx^m}{dt}\right) dt,$$

where F has the first-degree homogeneity with respect to $\dfrac{dx^i}{dt}$. Our convention for indices is as follows: Latin indices run from 1 to m (except m). Greek indices run from 1 to m. Greek indices with bar run from 1 to $m-1$. A Finsler manifold M has a tangent bundle $\pi\colon TM \to M$. From TM we obtain the projectivised tangent bundle of M, PTM, by identifying the non-zero vectors differing from each other by a real factor, Geometrically PTM is the space of line elements on M. Then a non-zero tangent vector can be expressed as

$$X = y^i \partial_{x^i} \quad (y^i \text{ not all zero}),$$

where we shall write shortly ∂_{x^i} (resp. ∂_{y^i}) instead of $\dfrac{\partial}{\partial x^i}$ (resp. $\dfrac{\partial}{\partial y^i}$). The x^i, y^i are local coordinates on TM. They are also local coordinates on PTM with y^i being

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Received: December 03, 2008.
2000 *Mathematics Subject Classification*: 53C60, 53D10.
Key words and phrases: Finsler manifold (space), the projectivised tangent bundle, Chern-Rund connection, conformally flat, Sasaki type metric.

homogeneous coordinates (determined up to a real factor). We can consider PTM as the base manifold of the vector bundle p^*TM, pulled back with the canonical projection map $p\colon PTM \to M$ defined by $p(x^i, y^i) = (x^i)$. The fibers of p^*TM are the vector spaces of dimension m and the base manifold PTM is of dimension $2m-1$ (see [4] and [6]). From now on $f_{y^i}, f_{y^i y^j}, \ldots$ etc. denote the partial derivative(s) of a smooth function f with respect to the coordinates y^i. Adopt a similar notation for the partial derivatives with respect to the coordinates x^i. From the first degree homogeneity of F, we have

$$y^i F_{y^i} = F$$

and

$$y^i F_{y^i y^j} = 0.$$

A differential form on PTM can be represented as one on TM provided the latter is invariant under rescaling in the y^i and yields zero when contracted with $y^i \partial_{y^i}$.

Then the Hilbert form

$$\omega = F_{y^i} dx^i$$

is intrinsically defined on PTM. Let

$$e_\alpha = {u_\alpha}^j \partial_{x^j}$$

be an orthonormal frame field on the bundle p^*TM, and

$$\omega^\alpha = {v^\alpha}_k dx^k$$

be its dual coframe, so that

$$(e_\alpha, e_\beta) = {u_\alpha}^\ell g_{\ell k} {u_\beta}^k = \delta_{\alpha\beta} \tag{1.1}$$

and

$$(e_\alpha, \omega^\beta) = {\delta^\beta}_\alpha. \tag{1.2}$$

(1.1) means the orthonormality condition with respect to the Finsler metric (positive definite).

$$G = g_{ij} dx^i \otimes dx^j = \left(\frac{1}{2}F^2\right)_{y^i y^j} dx^i \otimes dx^j$$

$$= (F F_{y^i y^j} + F_{y^i} F_{y^j}) dx^i \otimes dx^j$$

defined intrinsically on PTM, and (1.2) is the duality condition, which is equivalent

$${v^\beta}_k {u_\alpha}^k = {\delta^\beta}_\alpha.$$

We now distinguish the global section

$$e_m = \frac{y^i}{F}\partial_{x^i} := \ell^i \partial_{x^i}$$

and

$$\omega^m = F_{y^i} dx^i = \omega.$$

Then, taking the exterior derivative Hilbert form ω^m on PTM, we have ([6])

$$d\omega^m = \omega^{\overline{\alpha}} \wedge \omega_{\overline{\alpha}}{}^m, \tag{1.3}$$

where $\omega_{\overline{\alpha}}{}^m$ is

$$\omega_{\overline{\alpha}}{}^m = -u_{\overline{\alpha}}{}^i F_{y^i y^j} dy^j + \frac{u_{\overline{\alpha}}{}^i}{F}(F_{x^i} - y^j F_{y^i x^j})\omega^m$$
$$+ u_{\overline{\alpha}}{}^i u_{\overline{\beta}}{}^j F_{x^i y^j} \omega^{\overline{\beta}} + \lambda_{\overline{\alpha}\overline{\beta}} \omega^{\overline{\beta}}$$

(see [6] about $\lambda_{\overline{\alpha}\overline{\beta}}$).

Define $N^i{}_j$ and δy^j as follows:

$$N^i{}_j = \frac{1}{F} G^i_{y^j}, \quad \delta y^j = \frac{dy^j}{F} + N^j{}_k dx^k,$$

where G^i denotes

$$G^i = g^{i\ell}\left\{ y^s \left(\frac{1}{2}F^2\right)_{y^\ell x^s} - \left(\frac{1}{2}F^2\right)_{x^\ell} \right\}.$$

Moreover, we see that the dual orthonormal vectors in $T(TM\backslash 0)$ to the basis set $\omega^\alpha = v^\alpha{}_j dx^j$ and $\omega_m{}^\alpha = v^\alpha{}_j \delta y^j$ in $T^*(TM\backslash 0)$ are given by

$$\widehat{e}_\alpha = u_\alpha{}^j \delta_{x^j} \tag{1.4}$$

and

$$\widehat{e}_{m+\alpha} = u_\alpha{}^j \delta_{y^j}, \tag{1.5}$$

where

$$\delta_{x^i} := \partial_{x^i} - F N^j{}_i \partial_{y^j}$$

and

$$\delta_{y^i} := F\partial_{y^i}.$$

The set $\{\delta_{x^i}, \delta_{y^i}\}$ is naturally dual to the set $\{dx^i, \delta y^i\}$, and these form local bases for $T(TM\backslash\{0\})$ and $T^*(TM\backslash\{0\})$, respectively.

On the other hand, concerning the connection on M, S. S. Chern ([4] and [6]) proved the following theorem, however, afterwards, M. Anastasiei [1] proved that the Rund connection coincides with the Chern connection. So we have the following:

Theorem 1.1 (Chern-Rund). *There exists a torsion-free and an almost metric-comparability linear connection $p^*TM \to PTM$, that is the Chern-Rund connection*

$$D\colon \Gamma(p^*TM) \to \Gamma(p^*TM \otimes PTM),$$

given by $De_\alpha = \omega_\alpha{}^\beta e_\beta$, $\omega_m{}^m = 0$, that is $d\omega^\alpha = \omega^\beta \wedge \omega_\beta{}^\alpha$ and

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = -2A_{\alpha\beta\gamma}\omega^\gamma_m. \tag{1.6}$$

In particular

$$\omega_\alpha{}^m + \omega_m{}^\alpha = 0, \tag{1.7}$$

where $\omega_{\alpha\beta} = {\omega_\alpha}^\gamma \delta_{\gamma\beta}$ *and the Cartan tensor* $A = A_{\alpha\beta\gamma}\omega^\alpha \otimes \omega^\beta \otimes \omega^\gamma$ *is given by*

$$A_{\alpha\beta\gamma} = \frac{F}{2}\left(\frac{1}{2}F^2\right)_{y^i y^j yk} {u_\alpha}^i {u_\beta}^j {u_\gamma}^k.$$

Here we define the Chern-Rund connection in natural coordinates as follows:

$$D\colon \Gamma(p^*TM) \to \Gamma(p^*TM \otimes T^*(TM\backslash 0))$$

given by

$$D\partial_{x^i} = {\omega_i}^j \partial_{x^j},$$

where ${\omega_i}^j$ are the components of the connection matrix in natural coordinates. Since the Chern-Rund connection is torsion-free, we can see that (see [4] and [6])

$$dx^i \wedge {\omega_i}^j = 0, \tag{1.8}$$

which is equivalent to the torsion-free condition of the Chern-Rund connection in natural coordinates (see [3], [7], [8] and [9] for another connections of Finsler Geometry, and [2], [3] and [8] for the interesting application of Finsler Geometry). Wedge product of ${\omega_i}^j$ and dx^i is zero in (1.8), so they are linearly dependent. We can write ${\omega_i}^j$ in terms of dx^i as

$${\omega_i}^j = {\Gamma^j}_{i\ell} dx^\ell,$$

where the quantities

$${\Gamma^i}_{jk} = \frac{g^{is}}{2}\left(\delta_{x^k} g_{sj} - \delta_{x^s} g_{jk} + \delta_{x^j} g_{ks}\right)$$

are called the Christoffel symbols of the first. Then we obtain

$${\Gamma^i}_{jk} \ell^j = {N^i}_k. \tag{1.9}$$

Here we have the following lemma:

Lemma 1.1. *The dual orthonormal vectors* $($*in* $T(PTM))$ *to the basis set* $\omega^\alpha = {v^\alpha}_j dx^j$ *and* ${\omega_m}^{\overline{\alpha}} = {v^{\overline{\alpha}}}_j \delta y^j$ $($*in* $T^*(PTM))$ *are given by*

$$\widetilde{e}_\alpha = {u_\alpha}^j \partial_{x^j} \tag{1.10}$$

and

$$\widetilde{e}_{m+\overline{\alpha}} = {u_{\overline{\alpha}}}^j \delta_{y^j}. \tag{1.11}$$

Proof. By using the Cartan formula, we obtain the following Lie bracket (cf. [4]):

$$\left[\delta_{x^k}, \delta_{y^\ell}\right] = \left\{ {\dot{A}^i}_{k\ell} + \frac{\ell^i}{F}(FF_{y^k})_{x^\ell} - \ell^i N_{k\ell} \right\} \delta_{y^i}, \tag{1.12}$$

where the quantities ${\dot{A}^i}_{k\ell}$ are

$${\dot{A}^i}_{k\ell} := \left(\delta_{x^s} {A^i}_{k\ell} + {A^h}_{k\ell}{\Gamma^i}_{hs} - {A^i}_{h\ell}{\Gamma^h}_{ks} - {A^i}_{kh}{\Gamma^h}_{\ell s} \right) \ell^s.$$

On the other hand, by straightforward calculations we obtain

$$\left[\delta_{x^k}, \delta_{y^\ell}\right] = \frac{1}{2}{G^i}_{y^k y^\ell} \delta_{y^i} = \left\{ {\dot{A}^i}_{k\ell} + {\Gamma^i}_{k\ell} \right\} \delta_{y^i}. \tag{1.13}$$

On PTM, there are the quantities which are homogeneous of degree zero in the y^i. Let f be a smooth function on PTM. Using the Euler's theorem, we have

$$\ell^i \delta_{y^i}(f) = y^i f_{y^i} = 0. \tag{1.14}$$

From (1.12) and (1.14), it follows that

$$\begin{aligned}\left[\delta_{x^k}, \delta_{y^\ell}\right](f) &= \left\{ \dot{A}^i{}_{k\ell} + \frac{\ell^i}{F}(FF_{y^k})_{x^\ell} - \ell^i N_{k\ell} \right\} \delta_{y^i}(f) \\ &= \dot{A}^i{}_{k\ell} \delta_{y^i}(f). \end{aligned}\tag{1.15}$$

On the other hand, by (1.13), we have

$$\left[\delta_{x^k}, \delta_{y^\ell}\right](f) = \dot{A}^i{}_{k\ell}\delta_{y^i}(f) + \Gamma^i{}_{k\ell}\delta_{y^i}(f). \tag{1.16}$$

(1.15) and (1.16) imply

$$\Gamma^i{}_{k\ell}\delta_{y^i}(f) = 0. \tag{1.17}$$

By (1.9) and (1.17), we obtain

$$N^i{}_j \delta_{y^i}(f) = \ell^k \Gamma^i{}_{kj} \delta_{y^i}(f) = 0. \tag{1.18}$$

Hence, by (1.4), (1.5) and (1.18), we have (1.10) and (1.11). □

Here, on the manifold PTM, we consider a natural Riemannian metric (a Sasaki type metric on $TM \backslash \{0\}$)

$$g^s = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j. \tag{1.19}$$

For $\left\{\widetilde{e}_\alpha(\text{resp. } \omega^\alpha),\ \widetilde{e}_{m+\overline{\alpha}}(\text{resp. } \omega_m^{\overline{\alpha}}) \text{ in } T(PTM) \text{ (resp. } T^*(PTM))\right\}$, we can rewrite (1.19) as

$$g^s = \delta_{\alpha\beta}\omega^\alpha \otimes \omega^\beta + \delta_{m+\overline{\alpha}\, m+\overline{\beta}}\omega_m{}^{\overline{\alpha}} \otimes \omega_m{}^{\overline{\beta}}. \tag{1.20}$$

Concerning the orthonormal frame field $\{\widetilde{e}_\alpha, \widetilde{e}_{m+\overline{\alpha}}\}$ on PTM in Lemma 1.1, we have the following lemma:

Lemma 1.2. *The Lie brackets for the orthonormal vectors* $\widetilde{e}_\alpha$, $\widetilde{e}_{m+\overline{\alpha}}$ *are as follows:*

$$\left[\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{\overline{\beta}}\right] = \left(\omega_{\overline{\beta}}{}^{\gamma}\left(\widetilde{e}_{\overline{\alpha}}\right) - \omega_{\overline{\alpha}}{}^{\gamma}\left(\widetilde{e}_{\overline{\beta}}\right)\right)\widetilde{e}_\gamma, \tag{1.21}$$

$$\left[\widetilde{e}_\alpha, \widetilde{e}_{m+\overline{\beta}}\right] = \omega_{\overline{\beta}}{}^{\overline{\gamma}}(e_\alpha)\widetilde{e}_{m+\overline{\gamma}} - \omega_\alpha{}^\gamma(\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_\gamma, \tag{1.22}$$

$$\left[\widetilde{e}_{m+\overline{\alpha}}, \widetilde{e}_{m+\overline{\beta}}\right] = \left(\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_{m+\overline{\alpha}}) - \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_{m+\overline{\beta}})\right)\widetilde{e}_{m+\overline{\gamma}}, \tag{1.23}$$

in particular,

$$\left[\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_m\right] = -\omega_{\overline{\alpha}}{}^{\overline{\gamma}}\left(\widetilde{e}_m\right)\widetilde{e}_{\overline{\gamma}} \tag{1.24}$$

and

$$\left[\widetilde{e}_m, \widetilde{e}_{m+\overline{\alpha}}\right] = \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)\widetilde{e}_{m+\overline{\gamma}} - \widetilde{e}_{\overline{\alpha}}. \tag{1.25}$$

Moreover the following formulas hold good:

$$\nabla_{\widetilde{e}_{m+\overline{\alpha}}}\widetilde{e}_{m+\overline{\beta}} = \omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{m+\overline{\gamma}} + A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\gamma}}, \tag{1.26}$$

$$\nabla_{\widetilde{e}_{m+\overline{\alpha}}}\widetilde{e}_{\beta} = {\omega_{\beta}}^{\gamma}(\widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{\gamma} + {A^{\gamma}}_{\overline{\alpha}\beta}\widetilde{e}_{\gamma}, \tag{1.27}$$

$$\nabla_{\widetilde{e}_{\alpha}}\widetilde{e}_{m+\overline{\beta}} = {A^{\gamma}}_{\alpha\overline{\beta}}\widetilde{e}_{\gamma} + {\omega_{\overline{\beta}}}^{\overline{\gamma}}(\widetilde{e}_{\alpha})\widetilde{e}_{m+\overline{\gamma}}, \tag{1.28}$$

$$\nabla_{\widetilde{e}_{\alpha}}\widetilde{e}_{\beta} = {\omega_{\beta}}^{\gamma}(\widetilde{e}_{\alpha})\widetilde{e}_{\gamma} - {A^{\overline{\gamma}}}_{\alpha\beta}\widetilde{e}_{m+\overline{\gamma}}, \tag{1.29}$$

where ∇ is the Levi-Civita connection on PTM with respect to g^s.

Proof. Let f be a smooth function on PTM. By the definition of Lie bracket and

$${\omega_{\alpha}}^{\beta} = {v^{\beta}}_{i}(d{u_{\alpha}}^{i} + {u_{\alpha}}^{j}{\omega_{j}}^{i}) = {v^{\beta}}_{i}(d{u_{\alpha}}^{i} + {u_{\alpha}}^{j}{\Gamma^{i}}_{jk}dx^{k}),$$

we get

$$\begin{aligned}
\left[\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{\overline{\beta}}\right](f) &= \left[{u_{\overline{\alpha}}}^{i}\partial_{x^i}, {u_{\overline{\beta}}}^{j}\partial_{x^j}\right](f) \\
&= {u_{\overline{\alpha}}}^{i}{u_{\overline{\beta}}}^{j}\left[\partial_{x^i}, \partial_{x^j}\right](f) + {u_{\overline{\alpha}}}^{i}(\partial_{x^i}{u_{\overline{\beta}}}^{j})\partial_{x^j}(f) - {u_{\overline{\beta}}}^{j}(\partial_{x^j}{u_{\overline{\alpha}}}^{i})\partial_{x^i}(f) \\
&= \left({u_{\overline{\alpha}}}^{j}(\partial_{x^j}{u_{\overline{\beta}}}^{i}) - {u_{\overline{\beta}}}^{j}(\partial_{x^j}{u_{\overline{\alpha}}}^{i})\right)\partial_{x^i}(f) \\
&= \left({u_{\gamma}}^{i}{\omega_{\overline{\beta}}}^{\gamma}(\widetilde{e}_{\overline{\alpha}}) - {u_{\gamma}}^{i}{\omega_{\overline{\alpha}}}^{\gamma}\left(\widetilde{e}_{\overline{\beta}}\right)\right){v^{\delta}}_{i}\widetilde{e}_{\delta}(f) \\
&= \left({\omega_{\overline{\beta}}}^{\gamma}(\widetilde{e}_{\overline{\alpha}}) - {\omega_{\overline{\alpha}}}^{\gamma}\left(\widetilde{e}_{\overline{\beta}}\right)\right)\widetilde{e}_{\gamma}(f),
\end{aligned}$$

from which (1.21) holds. Similarly, we have (1.22) and (1.23).

Using (1.6), (1.22) and (1.23), we obtain

$$\begin{aligned}
2g^s(\nabla_{\widetilde{e}_{m+\overline{\alpha}}}\widetilde{e}_{m+\overline{\beta}}, \widetilde{e}_{\gamma}) &= g^s([\widetilde{e}_{m+\overline{\alpha}}, \widetilde{e}_{m+\overline{\beta}}], \widetilde{e}_{\gamma}) - g^s([\widetilde{e}_{m+\overline{\beta}}, \widetilde{e}_{\gamma}], \widetilde{e}_{m+\overline{\alpha}}) \\
&\quad + g^s([\widetilde{e}_{\gamma}, \widetilde{e}_{m+\overline{\alpha}}], \widetilde{e}_{m+\overline{\beta}}) \\
&= \omega_{\overline{\beta}\overline{\alpha}}(\widetilde{e}_{\gamma}) + \omega_{\overline{\alpha}\overline{\beta}}(\widetilde{e}_{\gamma}) = -2A_{\overline{\alpha}\overline{\beta}\delta}{\omega_{m}}^{\delta}(\widetilde{e}_{\gamma}) = 0
\end{aligned}$$

and

$$\begin{aligned}
2g^s(\nabla_{\widetilde{e}_{m+\overline{\alpha}}}\widetilde{e}_{m+\overline{\beta}}, \widetilde{e}_{m+\overline{\gamma}}) &= g^s([\widetilde{e}_{m+\overline{\alpha}}, \widetilde{e}_{m+\overline{\beta}}], \widetilde{e}_{m+\overline{\gamma}}) - g^s([\widetilde{e}_{m+\overline{\beta}}, \widetilde{e}_{m+\overline{\gamma}}], \widetilde{e}_{m+\overline{\alpha}}) \\
&\quad + g^s([\widetilde{e}_{m+\overline{\gamma}}, \widetilde{e}_{m+\overline{\alpha}}], \widetilde{e}_{m+\overline{\beta}}) \\
&= \omega_{\overline{\beta}\overline{\gamma}}(\widetilde{e}_{m+\overline{\alpha}}) - \omega_{\overline{\alpha}\,\overline{\gamma}}(\widetilde{e}_{m+\overline{\beta}}) - \omega_{\overline{\gamma}\overline{\alpha}}(\widetilde{e}_{m+\overline{\beta}}) \\
&\quad + \omega_{\overline{\beta}\overline{\alpha}}(\widetilde{e}_{m+\overline{\gamma}}) + \omega_{\overline{\alpha}\overline{\beta}}(\widetilde{e}_{m+\overline{\gamma}}) - \omega_{\overline{\gamma}\overline{\beta}}(\widetilde{e}_{m+\overline{\alpha}}) \\
&= 2\omega_{\overline{\beta}\overline{\gamma}}(\widetilde{e}_{m+\overline{\alpha}}) + 2A_{\overline{\alpha}\overline{\beta}\overline{\gamma}}.
\end{aligned}$$

Hence we have (1.26). Similarly we get (1.27), (1.28) and (1.29). □

Using Lemma 1.2, we have

$$\nabla_X\widetilde{e}_m = \sum_{\overline{\alpha}} g^s(X, \widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{\overline{\alpha}} \tag{1.30}$$

for any $X \in \chi(PTM)$, from which,

$$\begin{aligned}\widetilde{R}(X,Y)\widetilde{e}_m &= \nabla_X\nabla_Y\widetilde{e}_m - \nabla_Y\nabla_X\widetilde{e}_m - \nabla_{[X,Y]}\widetilde{e}_m\\ &= \sum_{\alpha}\sum_{\overline{\beta}}\{g^s(X,\widetilde{e}_\alpha)g^s(Y,\widetilde{e}_{m+\overline{\beta}})\\ &\qquad -g^s(Y,\widetilde{e}_\alpha)g^s(X,\widetilde{e}_{m+\overline{\beta}})\}A^{\overline{\gamma}}{}_{\alpha\overline{\beta}}\widetilde{e}_{\overline{\gamma}} \qquad (1.31)\end{aligned}$$

for any $X, Y \in \chi(PTM)$, where $\widetilde{R}$ is the curvature tensor on PTM with respect to g^s.

By being torsion-free, we have

$$\nabla_{\widetilde{e}_m}X = \sum_{\overline{\alpha}} g^s(X,\widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{\overline{\alpha}} + [\widetilde{e}_m, X] \qquad (1.32)$$

for any $X \in \chi(PTM)$.

Finally, the curvature form of the Chern-Rund connection is given by

$$\begin{aligned}\Omega_{\beta}{}^{\alpha} &= \frac{1}{2}R_{\beta}{}^{\alpha}{}_{\gamma\delta}\omega^{\gamma}\wedge\omega^{\delta} + P_{\beta}{}^{\alpha}{}_{\gamma\overline{\epsilon}}\omega^{\gamma}\wedge\omega_m^{\overline{\varepsilon}}\\ &= \frac{1}{2}R_{\beta}{}^{\alpha}{}_{\gamma\delta}\omega^{\gamma}\wedge\omega^{\delta} + P_{\beta}{}^{\alpha}{}_{\gamma\overline{\epsilon}}\omega^{\gamma}\wedge\omega^{m+\overline{\varepsilon}}, \qquad (1.33)\end{aligned}$$

where $R_{\beta}{}^{\alpha}{}_{\gamma\delta}$ and $P_{\beta}{}^{\alpha}{}_{\gamma\overline{\epsilon}}$ are the first and second Chern curvature tensors on M respectively.

2. THE CURVATURE TENSOR $\widetilde{R}$ ON PTM

For the curvature tensor $\widetilde{R}$ on PTM with respect to Sasaki type metric g^s, we have two following lemmas:

Lemma 2.1. *For the curvature tensor $\widetilde{R}$ on PTM with respect to g^s we have*

$$\begin{aligned}\widetilde{R}(\widetilde{e}_m,\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\beta}} &= R_{\overline{\beta}}{}^{\overline{\gamma}}{}_{m\overline{\alpha}}\widetilde{e}_{\overline{\gamma}} - \dot{A}^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\gamma}} - A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{m+\overline{\delta}}\\ &\quad +\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\alpha}\,\overline{\gamma}}\widetilde{e}_{m+\overline{\delta}} + \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\gamma}\overline{\beta}}\widetilde{e}_{m+\overline{\delta}}, \qquad (2.1)\end{aligned}$$

$$\begin{aligned}\widetilde{R}(\widetilde{e}_m,\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{m+\overline{\beta}} &= R_{\overline{\beta}}{}^{\overline{\gamma}}{}_{m\overline{\alpha}}\widetilde{e}_{m+\overline{\gamma}} + \dot{A}^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{\overline{\gamma}} + A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{\overline{\delta}}\\ &\quad -\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\alpha}\,\overline{\gamma}}\widetilde{e}_{\overline{\delta}} - \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\gamma}\overline{\beta}}\widetilde{e}_{\overline{\delta}}, \qquad (2.2)\end{aligned}$$

$$\begin{aligned}\widetilde{R}(\widetilde{e}_m,\widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{\overline{\beta}} &= P_{\overline{\beta}}{}^{\overline{\gamma}}{}_{m\overline{\alpha}}\widetilde{e}_{\overline{\gamma}} + \dot{A}^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{\overline{\gamma}} + A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{\overline{\delta}}\\ &\quad -\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\alpha}\,\overline{\gamma}}\widetilde{e}_{\overline{\delta}} - \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\gamma}\overline{\beta}}\widetilde{e}_{\overline{\delta}} - A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\gamma}}, \qquad (2.3)\end{aligned}$$

$$\begin{aligned}\widetilde{R}(\widetilde{e}_m,\widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{m+\overline{\beta}} &= P_{\overline{\beta}}{}^{\overline{\gamma}}{}_{m\overline{\alpha}}\widetilde{e}_{m+\overline{\gamma}} + \dot{A}^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\gamma}} + A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{m+\overline{\delta}}\\ &\quad -\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\alpha}\,\overline{\gamma}}\widetilde{e}_{m+\overline{\delta}} - \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\gamma}\overline{\beta}}\widetilde{e}_{m+\overline{\delta}} + A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{\overline{\gamma}}. \qquad (2.4)\end{aligned}$$

Proof. We show (2.1). From (1.32), (1.29), (1.24), (1.25) and (1.33) we have

$$
\begin{aligned}
&\widetilde{R}(\widetilde{e}_m, \widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\beta}} = \\
&\quad = \nabla_{\widetilde{e}_m}\nabla_{\widetilde{e}_{\overline{\alpha}}}\widetilde{e}_{\overline{\beta}} - \nabla_{\widetilde{e}_{\overline{\alpha}}}\nabla_{\widetilde{e}_m}\widetilde{e}_{\overline{\beta}} - \nabla_{[\widetilde{e}_m, \widetilde{e}_{\overline{\alpha}}]}\widetilde{e}_{\overline{\beta}} \\
&\quad = \sum_{\overline{\gamma}} g^s(\nabla_{\widetilde{e}_{\overline{\alpha}}}\widetilde{e}_{\overline{\beta}}, \widetilde{e}_{m+\overline{\gamma}})\widetilde{e}_{\overline{\gamma}} + [\widetilde{e}_m, \nabla_{\widetilde{e}_{\overline{\alpha}}}\widetilde{e}_{\overline{\beta}}] \\
&\qquad -\nabla_{\widetilde{e}_{\overline{\alpha}}}\left\{\sum_{\overline{\gamma}} g^s(\widetilde{e}_{\overline{\beta}}, \widetilde{e}_{m+\overline{\gamma}})\widetilde{e}_{\overline{\gamma}} + [\widetilde{e}_m, \widetilde{e}_{\overline{\beta}}]\right\} - \nabla_{[\widetilde{e}_m, \widetilde{e}_{\overline{\alpha}}]}\widetilde{e}_{\overline{\beta}} \\
&\quad = -A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{\overline{\gamma}} + \left[\widetilde{e}_m, \omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\gamma}} - A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\gamma}}\right] \\
&\qquad -\nabla_{\widetilde{e}_{\overline{\alpha}}}(\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)\widetilde{e}_{\overline{\gamma}}) - \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)\nabla_{\widetilde{e}_{\overline{\gamma}}}\widetilde{e}_{\overline{\beta}} \\
&\quad = -A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{\overline{\gamma}} + \widetilde{e}_m(\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_{\overline{\alpha}}))\widetilde{e}_{\overline{\gamma}} - \widetilde{e}_m(A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}} + \omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_{\overline{\alpha}})[\widetilde{e}_m, \widetilde{e}_{\overline{\gamma}}] \\
&\qquad -A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}[\widetilde{e}_m, \widetilde{e}_{m+\overline{\gamma}}] - \widetilde{e}_{\overline{\alpha}}(\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m))\widetilde{e}_{\overline{\gamma}} - \omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)\nabla_{\widetilde{e}_{\overline{\alpha}}}\widetilde{e}_{\overline{\gamma}} - \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)\nabla_{\widetilde{e}_{\overline{\gamma}}}\widetilde{e}_{\overline{\beta}} \\
&\quad = -A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{\overline{\gamma}} + \widetilde{e}_m(\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_{\overline{\alpha}}))\widetilde{e}_{\overline{\gamma}} - \dot{A}^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\gamma}} + \omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{\overline{\delta}} \\
&\qquad -A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{m+\overline{\delta}} + A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{\overline{\gamma}} - \widetilde{e}_{\overline{\alpha}}(\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m))\widetilde{e}_{\overline{\gamma}} - \omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\delta}} \\
&\qquad +\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\alpha}\,\overline{\gamma}}\widetilde{e}_{m+\overline{\delta}} - \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_m)\omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\gamma}})\widetilde{e}_{\overline{\delta}} + \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\gamma}\overline{\beta}}\widetilde{e}_{m+\overline{\delta}} \\
&\quad = d\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m, \widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\gamma}} + \omega_{\overline{\beta}}{}^{\overline{\gamma}}([\widetilde{e}_m, \widetilde{e}_{\overline{\alpha}}])\widetilde{e}_{\overline{\gamma}} - \dot{A}^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\gamma}} + \omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{\overline{\delta}} \\
&\qquad -A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{m+\overline{\delta}} - \omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\delta}} \\
&\qquad +\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\alpha}\,\overline{\gamma}}\widetilde{e}_{m+\overline{\delta}} - \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)\omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\gamma}})\widetilde{e}_{\overline{\delta}} + \omega^{\overline{\gamma}}_{\overline{\alpha}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\gamma}\overline{\beta}}\widetilde{e}_{m+\overline{\delta}} \\
&\quad = \Omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m, \widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\gamma}} + \omega_{\overline{\beta}}{}^{\epsilon} \wedge \omega_{\epsilon}{}^{\overline{\gamma}}(\widetilde{e}_m, \widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\gamma}} - \dot{A}^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\gamma}} + \omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{\overline{\delta}} \\
&\qquad -A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{m+\overline{\delta}} - \omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\delta}} \\
&\qquad +\omega^{\overline{\gamma}}_{\overline{\beta}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\alpha\gamma}}\widetilde{e}_{m+\overline{\delta}} + \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\gamma}\overline{\beta}}\widetilde{e}_{m+\overline{\delta}} \\
&\quad = \frac{1}{2}R_{\overline{\beta}}{}^{\overline{\gamma}}{}_{\epsilon\delta}\ \omega^{\epsilon} \wedge \omega^{\delta}(\widetilde{e}_m, \widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\gamma}} + P_{\overline{\beta}}{}^{\overline{\gamma}}{}_{\epsilon\delta}\ \omega^{\epsilon} \wedge \omega^{\delta}(\widetilde{e}_m, \widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\gamma}} - \dot{A}^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\gamma}} \\
&\qquad -A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{m+\overline{\delta}} + \omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\alpha}\,\overline{\gamma}}\widetilde{e}_{m+\overline{\delta}} + \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\gamma}\overline{\beta}}\widetilde{e}_{m+\overline{\delta}} \\
&\quad = R_{\overline{\beta}}{}^{\overline{\gamma}}{}_{m\overline{\alpha}}\widetilde{e}_{\overline{\gamma}} - \dot{A}^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\gamma}} - A^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_m)\widetilde{e}_{m+\overline{\delta}} \\
&\qquad +\omega_{\overline{\beta}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\alpha}\,\overline{\gamma}}\widetilde{e}_{m+\overline{\delta}} + \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(\widetilde{e}_m)A^{\overline{\delta}}{}_{\overline{\gamma}\overline{\beta}}\widetilde{e}_{m+\overline{\delta}}.
\end{aligned}
$$

Similarly we get (2.2),(2.3) and (2.4). □

Lemma 2.2. *For the curvature tensor* $\widetilde{R}$ *on* PTM *with respect to* g^s *we have*

$$
\begin{aligned}
&\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\gamma}} =\\
&\quad = R_{\overline{\gamma}}{}^{\overline{\delta}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{\overline{\delta}} - \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}} - \widetilde{e}_{\overline{\alpha}}(A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}})\widetilde{e}_{m+\overline{\delta}} - A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{\overline{\nu}}\\
&\qquad - A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{m+\overline{\nu}} + \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}} + \widetilde{e}_{\overline{\beta}}(A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}})\widetilde{e}_{m+\overline{\delta}} + A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{\overline{\nu}}\\
&\qquad + A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\nu}} + \omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{m+\overline{\nu}} - \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{m+\overline{\nu}},
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
&\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}} =\\
&\quad = R_{\overline{\gamma}}{}^{\overline{\delta}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\delta}} + \widetilde{e}_{\overline{\alpha}}(A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}})\widetilde{e}_{\overline{\delta}} + A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\nu}} - A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}} - \widetilde{e}_{\overline{\beta}}(A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}})\widetilde{e}_{\overline{\delta}}\\
&\qquad - A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\nu}} + A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}} - \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{\overline{\nu}} + \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{\overline{\nu}}\\
&\qquad - \omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{\overline{\nu}} + \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{\overline{\nu}},
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
&\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{\overline{\gamma}} =\\
&\quad = P_{\overline{\gamma}}{}^{\overline{\delta}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{\overline{\delta}} - \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{m+\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}} + \widetilde{e}_{\overline{\alpha}}(A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}})\widetilde{e}_{\overline{\delta}} + A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\nu}}\\
&\qquad - A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}} - \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{\overline{\nu}} + \widetilde{e}_{m+\overline{\beta}}(A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}})\widetilde{e}_{m+\overline{\delta}} + A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\nu}}\\
&\qquad + A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}} - \omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{\overline{\nu}} - \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{m+\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{m+\overline{\nu}},
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}} =\\
&\qquad\quad = P_{\overline{\gamma}}{}^{\overline{\delta}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\delta}} + \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{m+\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{\overline{\nu}} + \widetilde{e}_{\overline{\alpha}}(A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}})\widetilde{e}_{m+\overline{\delta}} + A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{\overline{\nu}}\\
&\qquad\qquad + A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{m+\overline{\nu}} - \widetilde{e}_{m+\overline{\beta}}(A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}})\widetilde{e}_{\overline{\delta}} - A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{\overline{\nu}}\\
&\qquad\qquad + A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}\delta_{\overline{\beta}\overline{\delta}}\widetilde{e}_{m} - A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{\overline{\nu}} - \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}}\\
&\qquad\qquad - \omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{m+\overline{\nu}} + \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{m+\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{\overline{\nu}},
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{\overline{\gamma}} =\\
&\quad = \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{m+\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{\overline{\nu}} - \delta_{\overline{\beta}\overline{\gamma}}\widetilde{e}_{\overline{\alpha}} + \widetilde{e}_{m+\overline{\alpha}}(A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}})\widetilde{e}_{\overline{\delta}} + A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{\overline{\nu}} + A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{\overline{\nu}}\\
&\qquad - \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{m+\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{\overline{\nu}} + \delta_{\overline{\alpha}\overline{\gamma}}\widetilde{e}_{\overline{\beta}} - \widetilde{e}_{m+\overline{\beta}}(A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}})\widetilde{e}_{\overline{\delta}} - A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{\overline{\nu}}\\
&\qquad - A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{\overline{\nu}} - \omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{m+\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{\overline{\nu}} + \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{m+\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{\overline{\nu}},
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}} =\\
&= \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{m+\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}} + \widetilde{e}_{m+\overline{\alpha}}(A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}})\widetilde{e}_{m+\overline{\delta}} + A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{m+\overline{\nu}} + A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}}\\
&\quad - \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{m+\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}} - \widetilde{e}_{m+\overline{\beta}}(A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}})\widetilde{e}_{m+\overline{\delta}} - A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\nu}}\\
&\quad - A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}} - \omega^{\overline{\delta}}_{\overline{\beta}}(\widetilde{e}_{m+\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{m+\overline{\nu}} + \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{m+\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\delta}\overline{\gamma}}\widetilde{e}_{m+\overline{\nu}}.
\end{aligned}
\tag{2.10}
$$

Proof. First we show (2.5). From (1.28) and (1.29), we have

$$\begin{aligned}\nabla_{\widetilde{e}_{\overline{\alpha}}}\nabla_{\widetilde{e}_{\overline{\beta}}}\widetilde{e}_{\overline{\gamma}} &= \nabla_{\widetilde{e}_{\overline{\alpha}}}(\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\delta}} - A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}\widetilde{e}_{m+\overline{\delta}})\\ &= \widetilde{e}_{\overline{\alpha}}(\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}}))\widetilde{e}_{\overline{\delta}} + \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})\nabla_{\widetilde{e}_{\overline{\alpha}}}\widetilde{e}_{\overline{\delta}} - \widetilde{e}_{\overline{\alpha}}(A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}})\widetilde{e}_{m+\overline{\delta}} - A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}\nabla_{\widetilde{e}_{\overline{\alpha}}}\widetilde{e}_{m+\overline{\delta}}\\ &= \widetilde{e}_{\overline{\alpha}}(\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}}))\widetilde{e}_{\overline{\delta}} + \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\nu}} - \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}}\\ &\quad -\widetilde{e}_{\overline{\alpha}}(A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}})\widetilde{e}_{m+\overline{\delta}} - A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{\overline{\nu}} - A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{m+\overline{\nu}},\end{aligned}\tag{2.11}$$

from which

$$\begin{aligned}-\nabla_{\widetilde{e}_{\overline{\beta}}}\nabla_{\widetilde{e}_{\overline{\alpha}}}\widetilde{e}_{\overline{\gamma}} &= -\widetilde{e}_{\overline{\beta}}(\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}}))\widetilde{e}_{\overline{\delta}} - \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\nu}} + \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}}\\ &\quad -\widetilde{e}_{\overline{\beta}}(A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}})\widetilde{e}_{m+\overline{\delta}} + A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{\overline{\nu}} + A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\nu}}.\end{aligned}\tag{2.12}$$

Moreover, from (1.21) and (1.29) we can see that

$$\begin{aligned}-\nabla_{[\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}}]}\widetilde{e}_{\overline{\gamma}} &= -(\omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}}) - \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}}))\nabla_{\widetilde{e}_{\overline{\delta}}}\widetilde{e}_{\overline{\gamma}}\\ &= -(\omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}}) - \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}}))(\omega_{\overline{\gamma}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\delta}})\widetilde{e}_{\overline{\nu}} - A^{\overline{\nu}}{}_{\delta\overline{\gamma}}\widetilde{e}_{m+\overline{\nu}}) = -\omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\gamma}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\delta}})\widetilde{e}_{\overline{\nu}}\\ &\quad +\omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})\omega_{\overline{\gamma}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\delta}})\widetilde{e}_{\overline{\nu}} + \omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\delta\overline{\gamma}}\widetilde{e}_{m+\overline{\nu}} - \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})A^{\overline{\nu}}{}_{\delta\overline{\gamma}}\widetilde{e}_{m+\overline{\nu}}.\end{aligned}\tag{2.13}$$

Here, we get, by (1.33),

$$\begin{aligned}&\widetilde{e}_{\overline{\alpha}}(\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}}))\widetilde{e}_{\overline{\delta}} - \widetilde{e}_{\overline{\beta}}(\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}}))\widetilde{e}_{\overline{\delta}} = d\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\delta}} + \omega_{\overline{\gamma}}{}^{\overline{\delta}}([\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}}])\widetilde{e}_{\overline{\delta}}\\ &\quad = \Omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\delta}} + \omega_{\overline{\gamma}}{}^{\epsilon}\wedge\omega_{\epsilon}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\delta}} + \omega_{\overline{\beta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\nu}})\widetilde{e}_{\overline{\delta}} - \omega_{\overline{\alpha}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\beta}})\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\nu}})\widetilde{e}_{\overline{\delta}}\\ &\quad = R_{\overline{\gamma}}{}^{\overline{\delta}}{}_{\overline{\alpha}\overline{\beta}} + \omega_{\overline{\gamma}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\nu}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\delta}} - \omega_{\overline{\gamma}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\beta}})\omega_{\overline{\nu}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\delta}}\\ &\qquad +\omega_{\overline{\beta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\nu}})\widetilde{e}_{\overline{\delta}} - \omega_{\overline{\alpha}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\beta}})\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\nu}})\widetilde{e}_{\overline{\delta}}.\end{aligned}\tag{2.14}$$

By means of (2.11), (2.12),(2.13) and (2.14), we obtain (2.5).

Similarly we prove (2.6).

$$\begin{aligned}\nabla_{\widetilde{e}_{\overline{\alpha}}}\nabla_{\widetilde{e}_{\overline{\beta}}}\widetilde{e}_{m+\overline{\gamma}} &= \widetilde{e}_{\overline{\alpha}}(A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}})\widetilde{e}_{\overline{\delta}} + A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\nu}} - A^{\overline{\delta}}{}_{\overline{\beta}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}}\\ &\quad +\widetilde{e}_{\overline{\alpha}}(\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\delta}} + \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})A^{\overline{\nu}}{}_{\overline{\alpha}\overline{\delta}}\widetilde{e}_{\overline{\nu}} + \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{m+\overline{\nu}},\end{aligned}\tag{2.15}$$

$$\begin{aligned}-\nabla_{\widetilde{e}_{\overline{\beta}}}\nabla_{\widetilde{e}_{\overline{\alpha}}}\widetilde{e}_{m+\overline{\gamma}} &= -\widetilde{e}_{\overline{\beta}}(A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}})\widetilde{e}_{\overline{\delta}} - A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\nu}} + A^{\overline{\delta}}{}_{\overline{\alpha}\overline{\gamma}}A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{m+\overline{\nu}} - \widetilde{e}_{\overline{\beta}}(\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}}))\widetilde{e}_{m+\overline{\delta}}\\ &\quad -\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\overline{\beta}\overline{\delta}}\widetilde{e}_{\overline{\nu}} - \omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\delta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\nu}},\end{aligned}\tag{2.16}$$

$$\begin{aligned}-\nabla_{[\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}}]}\widetilde{e}_{m+\overline{\gamma}} &= -\omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})A^{\overline{\nu}}{}_{\delta\overline{\gamma}}\widetilde{e}_{\overline{\nu}} + \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})A^{\overline{\nu}}{}_{\delta\overline{\gamma}}\widetilde{e}_{m+\overline{\nu}}\\ &= -\omega_{\overline{\beta}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\gamma}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\delta}})\widetilde{e}_{m+\overline{\nu}} + \omega_{\overline{\alpha}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})\omega_{\overline{\gamma}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\delta}})\widetilde{e}_{m+\overline{\nu}},\end{aligned}\tag{2.17}$$

$$\begin{aligned}
&\widetilde{e}_{\overline{\alpha}}(\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}}))\widetilde{e}_{m+\overline{\delta}} - \widetilde{e}_{\overline{\beta}}(\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}}))\widetilde{e}_{m+\overline{\delta}} = \\
&\qquad = R_{\overline{\gamma}}{}^{\overline{\delta}}{}_{\overline{\alpha}\overline{\beta}}\widetilde{e}_{m+\overline{\delta}} + \omega_{\overline{\gamma}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\nu}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\delta}} - \omega_{\overline{\gamma}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\beta}})\omega_{\overline{\nu}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{m+\overline{\delta}} \\
&\qquad\quad + \omega_{\overline{\beta}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\alpha}})\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\nu}})\widetilde{e}_{m+\overline{\delta}} - \omega_{\overline{\alpha}}{}^{\overline{\nu}}(\widetilde{e}_{\overline{\beta}})\omega_{\overline{\gamma}}{}^{\overline{\delta}}(\widetilde{e}_{\overline{\nu}})\widetilde{e}_{m+\overline{\delta}}.
\end{aligned} \tag{2.18}$$

By means of (2.15), (2.16), (2.17) and (2.18), we obtain (2.6). Similarly we get (2.7), (2.8), (2.9) and (2.10). □

3. MAIN THEOREM

First we have the following theorem.

Theorem 3.1. *If the projectivised tangent bundle PTM of an m-dimensional Finsler space (M, F) with the Chern-Rund connection has the Sasaki type metric g^s as a Riemannian manifold, then the base manifold (M, F) is Riemannian and the (first) curvature tensor R of (M, F) satisfies*

$$R_{\overline{\alpha}\overline{\beta}\overline{\gamma}\,\overline{\nu}} = \delta_{\overline{\alpha}\,\overline{\gamma}}\delta_{\overline{\beta}\overline{\nu}} - \delta_{\overline{\beta}\overline{\gamma}}\delta_{\overline{\alpha}\,\overline{\nu}} \quad \textit{and} \quad R_{\overline{\alpha}\overline{\beta}m\overline{\gamma}} = 0.$$

Moreover the curvature tensor $\widetilde{R}$ on PTM can be given by

$$\left.\begin{aligned}
&\widetilde{R}(\widetilde{e}_{\alpha}, \widetilde{e}_{\beta})\widetilde{e}_m = \widetilde{R}(\widetilde{e}_{\alpha}, \widetilde{e}_{m+\overline{\beta}})\widetilde{e}_m = \widetilde{R}(\widetilde{e}_{m+\overline{\alpha}}, \widetilde{e}_{m+\overline{\beta}})\widetilde{e}_m = 0,\\
&\widetilde{R}(\widetilde{e}_m, \widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\beta}} = \widetilde{R}(\widetilde{e}_m, \widetilde{e}_{\overline{\alpha}})\widetilde{e}_{m+\overline{\beta}} = 0,\\
&\widetilde{R}(\widetilde{e}_m, \widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{\overline{\beta}} = \widetilde{R}(\widetilde{e}_m, \widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{m+\overline{\beta}} = 0,\\
&\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}}, \widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}} = 0,\\
&\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{\overline{\gamma}} = \widetilde{R}(\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}} = 0,\\
&\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}} = \delta_{\overline{\alpha}\,\overline{\gamma}}\widetilde{e}_{m+\overline{\beta}} - \delta_{\overline{\beta}\overline{\gamma}}\widetilde{e}_{m+\overline{\alpha}},\\
&\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\gamma}} = \widetilde{R}(\widetilde{e}_{m+\overline{\alpha}}, \widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{\overline{\gamma}} = \delta_{\overline{\alpha}\,\overline{\gamma}}\widetilde{e}_{\overline{\beta}} - \delta_{\overline{\beta}\overline{\gamma}}\widetilde{e}_{\overline{\alpha}}.
\end{aligned}\right\} \tag{3.1}$$

Proof. By means of (2.8), we have

$$g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}}, \widetilde{e}_m) = A_{\overline{\beta}\overline{\alpha}\,\overline{\gamma}}. \tag{3.2}$$

On the other hand, from (1.31), it follows that

$$g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}}, \widetilde{e}_m) = -g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{m+\overline{\beta}})\widetilde{e}_m, \widetilde{e}_{m+\overline{\gamma}}) = g^s(A^{\overline{\epsilon}}{}_{\overline{\alpha}\,\overline{\gamma}}\widetilde{e}_{\overline{\epsilon}}, \widetilde{e}_{m+\overline{\gamma}}) = 0. \tag{3.3}$$

Hence we can see that (3.2) and (3.3) imply

$$A_{\alpha\beta\gamma} = 0, \tag{3.4}$$

that is, the Cartan tensor vanishes.

From

$$g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}}, \widetilde{e}_{m+\overline{\nu}}) = g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\gamma}}, \widetilde{e}_{m+\overline{\nu}})\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{\overline{\beta}}),$$

(2.6) and (2.9), we get

$$R_{\overline{\gamma}\,\overline{\nu}\,\overline{\alpha}\overline{\beta}} = -\delta_{\overline{\nu}\,\overline{\alpha}}\delta_{\overline{\gamma}\overline{\beta}} + \delta_{\overline{\gamma}\,\overline{\alpha}}\delta_{\overline{\nu}\overline{\beta}}. \tag{3.5}$$

Since we also have

$$g^s(\widetilde{R}(X,Y)\widetilde{e}_m, Z) = g^s(\widetilde{R}(\widetilde{e}_m, Z)X, Y),$$

by (1.31), (2.1) and (3.4), we obtain

$$R_{\overline{\beta}\overline{\gamma}m\overline{\alpha}} = 0. \tag{3.6}$$

From the Cartan tensor $A = 0$, Lemma 2.1, Lemma 2.2, (1.33), (3.5) and (3.6), it follows that (3.1). This completes the proof. □

Corollary 3.1. *In Theorem* 3.1*, we assume that* $m = 2$*, i.e.,* PTM *is a* 3*-dimensional Riemannian manifold with Sasaki type metric* g^s*. Then* PTM *is flat.*

Proof. From (3.1), it follows that the curvature tensor $\widetilde{R}$ on PTM vanishes. □

From Theorem 3.1, we obtain the following:

Lemma 3.1. *For any vector fields* $X, Y \in T(PTM)$*, the Ricci tensor* $\widetilde{Ric}$ *(Ricci operator* $\widetilde{Q}$*) and the scalar curvature tensor* $\widetilde{r}$ *with respect to* g^s *on* PTM *are as follows :*

$$g^s(\widetilde{Q}X, Y) := \widetilde{Ric}(X,Y) = \sum_{\overline{\alpha}}\sum_{\overline{\beta}}\sum_{\overline{\gamma}} g^s(X, \widetilde{e}_{\overline{\beta}})g^s(Y, \widetilde{e}_{\overline{\gamma}})\left\{\delta_{\overline{\alpha}\overline{\beta}}\delta_{\overline{\alpha}\,\overline{\gamma}} - \delta_{\overline{\beta}\overline{\gamma}}\right\}, \tag{3.7}$$

$$\widetilde{r} = -(m-1)(m-2) \qquad \textit{(non-positive constant).} \tag{3.8}$$

Proof. The Ricci tensor $\widetilde{Ric}(X,Y)$ is given by

$$\begin{aligned}\widetilde{Ric}(X,Y) = g^s(\widetilde{R}(\widetilde{e}_m, X)Y, \widetilde{e}_m) &+ \sum_{\overline{\alpha}} g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, X)Y, \widetilde{e}_{\overline{\alpha}}) \\ &+ \sum_{\overline{\alpha}} g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}}, X)Y, \widetilde{e}_{m+\overline{\alpha}}).\end{aligned} \tag{3.9}$$

So, we will calculate $g^s(\widetilde{R}(\widetilde{e}_m, X)Y, \widetilde{e}_m)$, $\sum_{\overline{\alpha}} g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, X)Y, \widetilde{e}_{\overline{\alpha}})$ and $\sum_{\overline{\alpha}} g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}}, X) Y, \widetilde{e}_{m+\overline{\alpha}})$. By (3.1), we have

$$g^s(\widetilde{R}(\widetilde{e}_m, X)Y, \widetilde{e}_m) = -g^s(\widetilde{R}(\widetilde{e}_m, X)\widetilde{e}_m, Y) = 0. \tag{3.10}$$

Next we calculate $\sum_{\overline{\alpha}} g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, X)Y, \widetilde{e}_{\overline{\alpha}})$. Putting

$$X = \sum_{\overline{\beta}} g^s(X, \widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\beta}} + g^s(X, \widetilde{e}_m)\widetilde{e}_m + \sum_{\overline{\beta}} g^s(X, \widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\beta}}$$

and

$$Y = \sum_{\overline{\gamma}} g^s(Y, \widetilde{e}_{\overline{\gamma}})\widetilde{e}_{\overline{\gamma}} + g^s(Y, \widetilde{e}_m)\widetilde{e}_m + \sum_{\overline{\gamma}} g^s(Y, \widetilde{e}_{m+\overline{\gamma}})\widetilde{e}_{m+\overline{\gamma}},$$

we get

$$\begin{aligned}
&\sum_{\overline{\alpha}} g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},X)Y,\widetilde{e}_{\overline{\alpha}}) = \sum_{\overline{\alpha},\overline{\beta},\overline{\gamma}} g^s(X,\widetilde{e}_{\overline{\beta}})g^s(Y,\widetilde{e}_{\overline{\gamma}})g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{\overline{\alpha}})\\
&\quad+\sum_{\overline{\alpha},\overline{\beta}} g^s(X,\widetilde{e}_{\overline{\beta}})g^s(Y,\widetilde{e}_m)g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_m,\widetilde{e}_{\overline{\alpha}}) + \sum_{\overline{\alpha},\overline{\beta},\overline{\gamma}} g^s(X,\widetilde{e}_{\overline{\beta}})g^s(Y,\widetilde{e}_{m+\overline{\gamma}})\\
&\quad\quad g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{\overline{\alpha}}) + \sum_{\overline{\alpha},\overline{\gamma}} g^s(X,\widetilde{e}_m)g^s(Y,\widetilde{e}_{\overline{\gamma}})g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{\overline{\alpha}})\\
&\quad+\sum_{\overline{\alpha}} g^s(X,\widetilde{e}_m)g^s(Y,\widetilde{e}_m)g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_m,\widetilde{e}_{\overline{\alpha}}) + \sum_{\overline{\alpha},\overline{\gamma}} g^s(X,\widetilde{e}_m)g^s(Y,\widetilde{e}_{m+\overline{\gamma}})\\
&\quad\quad g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{\overline{\alpha}}) + \sum_{\overline{\alpha},\overline{\beta},\overline{\gamma}} g^s(X,\widetilde{e}_{m+\overline{\beta}})g^s(Y,\widetilde{e}_{\overline{\gamma}})g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{\overline{\alpha}})\\
&\quad+\sum_{\overline{\alpha},\overline{\beta}} g^s(X,\widetilde{e}_{m+\overline{\beta}})g^s(Y,\widetilde{e}_m)g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_m,\widetilde{e}_{\overline{\alpha}})\\
&\quad+\sum_{\overline{\alpha},\overline{\beta},\overline{\gamma}} g^s(X,\widetilde{e}_{m+\overline{\beta}})g^s(Y,\widetilde{e}_{m+\overline{\gamma}})g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{\overline{\alpha}}).
\end{aligned}$$

Here, using (3.1), we have

$$\begin{aligned}
&g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{\overline{\alpha}}) = R_{\overline{\gamma}\,\overline{\alpha}\,\overline{\alpha}\overline{\beta}} = \delta_{\overline{\alpha}\overline{\beta}}\delta_{\overline{\alpha}\,\overline{\gamma}} - \delta_{\overline{\beta}\overline{\gamma}},\\
&g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_m,\widetilde{e}_{\overline{\alpha}}) = -g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m) = 0,\\
&g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{\overline{\alpha}}) = 0,\\
&g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{\overline{\alpha}}) = g^s(\widetilde{R}(\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m) = 0,\\
&g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_m,\widetilde{e}_{\overline{\alpha}}) = -g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m) = 0,\\
&g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{\overline{\alpha}}) = -g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{\overline{\alpha}})\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m) = 0,\\
&g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{\overline{\alpha}}) = 0,\\
&g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_m,\widetilde{e}_{\overline{\alpha}}) = -g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{\overline{\alpha}},\widetilde{e}_m) = 0,\\
&g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{\overline{\alpha}}) = 0,
\end{aligned}$$

so that, we get

$$\sum_{\overline{\alpha}} g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}},X)Y,\widetilde{e}_{\overline{\alpha}}) = \sum_{\overline{\alpha},\overline{\beta},\overline{\gamma}} g^s(X,\widetilde{e}_{\overline{\beta}})g^s(Y,\widetilde{e}_{\overline{\gamma}})\left\{\delta_{\overline{\alpha}\overline{\beta}}\delta_{\overline{\alpha}\,\overline{\gamma}} - \delta_{\overline{\beta}\overline{\gamma}}\right\}. \tag{3.11}$$

Similarly it follows that

$$\begin{aligned}
&\sum_{\overline{\alpha}} g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},X)Y,\widetilde{e}_{m+\overline{\alpha}}) = \sum_{\overline{\alpha},\overline{\beta},\overline{\gamma}} g^s(X,\widetilde{e}_{\overline{\beta}})g^s(Y,\widetilde{e}_{\overline{\gamma}})g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})\\
&\quad+\sum_{\overline{\alpha},\overline{\beta}} g^s(X,\widetilde{e}_{\overline{\beta}})g^s(Y,\widetilde{e}_m)g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_m,\widetilde{e}_{m+\overline{\alpha}}) + \sum_{\overline{\alpha},\overline{\beta},\overline{\gamma}} g^s(X,\widetilde{e}_{\overline{\beta}})\\
&\quad\quad g^s(Y,\widetilde{e}_{m+\overline{\gamma}})g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}}) + \sum_{\overline{\alpha},\overline{\gamma}} g^s(X,\widetilde{e}_m)g^s(Y,\widetilde{e}_{\overline{\gamma}})
\end{aligned}$$

$$\begin{aligned}
&g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})+\sum_{\overline{\alpha}}g^s(X,\widetilde{e}_m)g^s(Y,\widetilde{e}_m)g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_m,\widetilde{e}_{m+\overline{\alpha}})\\
&+\sum_{\overline{\alpha},\overline{\gamma}}g^s(X,\widetilde{e}_m)g^s(Y,\widetilde{e}_{m+\overline{\gamma}})g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})\\
&+\sum_{\overline{\alpha},\overline{\beta},\overline{\gamma}}g^s(X,\widetilde{e}_{m+\overline{\beta}})g^s(Y,\widetilde{e}_{\overline{\gamma}})g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})\\
&+\sum_{\overline{\alpha},\overline{\beta}}g^s(X,\widetilde{e}_{m+\overline{\beta}})g^s(Y,\widetilde{e}_m)g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_m,\widetilde{e}_{m+\overline{\alpha}})\\
&+\sum_{\overline{\alpha},\overline{\beta},\overline{\gamma}}g^s(X,\widetilde{e}_{m+\overline{\beta}})g^s(Y,\widetilde{e}_{m+\overline{\gamma}})g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}}).
\end{aligned}$$

Moreover, from (3.1), we see that

$$\begin{aligned}
&g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})=-g^s(\widetilde{R}(\widetilde{e}_{\overline{\beta}},\widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})=0,\\
&g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_m,\widetilde{e}_{m+\overline{\alpha}})=-g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_m)=0,\\
&g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})=-g^s(\widetilde{R}(\widetilde{e}_{\overline{\beta}},\widetilde{e}_{m+\overline{\alpha}})\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})=0,\\
&g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})=g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_m,\widetilde{e}_{m+\overline{\alpha}})\\
&\qquad=g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_m)\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})=g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})\\
&\qquad=g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_m,\widetilde{e}_{m+\overline{\alpha}})=0,\\
&g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},\widetilde{e}_{m+\overline{\beta}})\widetilde{e}_{m+\overline{\gamma}},\widetilde{e}_{m+\overline{\alpha}})=0,
\end{aligned}$$

from which

$$\sum_{\overline{\alpha}}g^s(\widetilde{R}(\widetilde{e}_{m+\overline{\alpha}},X)Y,\widetilde{e}_{m+\overline{\alpha}})=0. \tag{3.12}$$

Substituting (3.10) $\sim$ (3.12) in (3.9), we get (3.7). Finally, we calculate the Scalar curvature tensor $\widetilde{r}$ on PTM. $\widetilde{r}$ is given by, from (3.7),

$$\widetilde{r}=\sum_{\overline{\epsilon}}\widetilde{Ric}(\widetilde{e}_{\overline{\epsilon}},\widetilde{e}_{\overline{\epsilon}})+\sum_{\overline{\epsilon}}\widetilde{Ric}(\widetilde{e}_{m+\overline{\epsilon}},\widetilde{e}_{m+\overline{\epsilon}}).$$

From (3.7), it follows that

$$\begin{aligned}
\widetilde{r}&=\sum_{\overline{\epsilon}}\widetilde{Ric}(\widetilde{e}_{\overline{\epsilon}},\widetilde{e}_{\overline{\epsilon}})=\sum_{\overline{\epsilon},\overline{\alpha},\overline{\beta},\overline{\gamma}}g^s(\widetilde{e}_{\overline{\epsilon}},\widetilde{e}_{\overline{\beta}})g^s(\widetilde{e}_{\overline{\epsilon}},\widetilde{e}_{\overline{\gamma}})\left\{\delta_{\overline{\alpha}\overline{\beta}}\delta_{\overline{\alpha}\,\overline{\gamma}}-\delta_{\overline{\beta}\overline{\gamma}}\right\}\\
&=\sum_{\overline{\epsilon},\overline{\alpha}}\left\{(\delta_{\overline{\alpha}\,\overline{\epsilon}})^2-1\right\}=-(m-1)(m-2).
\end{aligned}$$

This is (3.8). $\square$

From Lemma 3.1 and Theorem 3.1, we have the following theorem:

Theorem 3.2. *If the projectivised tangent bundle PTM of an m-dimensional $(m>2)$ Finsler space (M,F) with the Chern-Rund connection has the Sasaki type metric g^s, then there does not exist a conformally flat PTM on (M,F).*

Proof. Assume that PTM is conformally flat. Then we have

$$\begin{aligned} g^s(\widetilde{R}(X,Y)Z,W) = \\ &= \frac{1}{2m-3}\Big\{g^s(\widetilde{Q}Y,Z)g^s(X,W) - g^s(\widetilde{Q}X,Z)g^s(Y,W) \\ &\quad + g^s(Y,Z)g^s(\widetilde{Q}X,Y) - g^s(X,Z)g^s(\widetilde{Q}Y,W)\Big\} \\ &\quad + \frac{\widetilde{r}}{(2m-2)(2m-3)}\left\{g^s(Y,Z)g^s(X,W) - g^s(X,Z)g^s(Y,W)\right\}. \end{aligned} \tag{3.13}$$

Putting $\widetilde{e}_{\overline{\alpha}}$, $\widetilde{e}_{\overline{\beta}}$, $\widetilde{e}_{m+\overline{\beta}}$ and $\widetilde{e}_{m+\overline{\alpha}}$ into X, Y, Z, and W in (3.13) (where $\overline{\alpha} \neq \overline{\beta}$), we get

$$g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\beta}}, \widetilde{e}_{m+\overline{\alpha}}) = 0. \tag{3.14}$$

On the other hand, from (3.1), we obtain

$$g^s(\widetilde{R}(\widetilde{e}_{\overline{\alpha}}, \widetilde{e}_{\overline{\beta}})\widetilde{e}_{m+\overline{\beta}}, \widetilde{e}_{m+\overline{\alpha}}) = -1. \tag{3.15}$$

(3.14) and (3.15) lead us to the contradiction. This completes the proof. □

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Utsunomiya University,
Department of Mechanical Eng.,
Yoto 7-1-2*, Utsunomiya-shi* 321-8585*, Japan*
E-mail address: `hsk-endo@cc.utsunomiya-u.ac.jp`

Tokoha Gakuen University,
Faculty of Education,
Sena 1-22-1*, Shizuoka-shi* 420-0911*, Japan*
E-mail address: `s-fueki@tokoha-u.ac.jp`