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A NOTE ON QUASI-LARGE SUBGROUPS IN ABELIAN GROUP RINGS

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ABSTRACT. We prove that if G is an abelian group with a p-subgroup H and R is a commutative unital ring of prime characteristic p without nilpotents, then $S(RG;H)$ is a quasi-large subgroup of $S(RG)$, the normed Sylow p-group in the group ring RG, if and only if H is a quasi-large subgroup of G_p. This finalizes a recent result due to the first author in An. St. Univ. "Al. I. Cuza", Iaşi - Mat., 2006.

Throughout this brief article, let G be an abelian group with p-component of torsion G_p and R a commutative ring with 1 (called also unital) of prime characteristic p with nil-radical $N(R)$. Traditionally, RG will denote the group ring of G over R with group $S(RG)$ consisting of all normalized invertible elements of orders which are powers of p. For a subgroup H of G, the letter $I(RG;H)$ denotes the augmentation ideal of RG with respect to H. When $H \leq G_p$, for a facilitating of the exposition, the expression $1 + I(RG;H)$ is designed as $S(RG;H)$; clearly $S(RG;H)$ is a p-group, that is, $S(RG;H) \leq S(RG)$. For all other unexplained exclusively notions and notations, we refer the reader to [4].

In [1], Benabdallah jointly with Wilson introduced the concept of *a quasi-large subgroup* of an abelian p-group and established the following important necessary and sufficient condition.

Criterion 1.1 (Benabdallah-Wilson, 1978)**.** *Suppose C is a subgroup of the abelian p-group A. Then C is quasi-large in A $\iff$ $\exists\ (m_i)_{i<\omega} \in \mathbb{N}$ such that $m_i < m_{i+1}$ and $A^{p^{m_i}}[p^i] \subset C$ for all $i < \omega$.*

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Remark 1.1. Although it is clear from the context, the group G in ([2], p. 201, Definition) and ([2], p. 202, Criterion) is p-primary.

In [2] we find sufficient conditions for $S(RG; H)$ to be quasi-large in $S(RG)$ using some results from [3]. The aim of the present short paper is to obtain necessary conditions for this subgroup to be quasi-large in $S(RG)$ and by summarizing both results to derive a suitable criterion for that property of $S(RG; H)$.

Before doing this, we need the following classical technicality, which is well-known but for completeness of the exposition and for the reader's convenience we provide a proof.

Lemma 1.1. *For $H \leq G_p$ the following equality holds*

$$G \cap S(RG; H) = H.$$

Proof. Clearly, the left hand-side contains the right hand-side. To verify the converse inclusion, given x in the intersection. Hence $x = \sum_{g \in G} r_g g \in G$, written in canonical form, where $r_g \in R$ and $\sum_{g \in aH} r_g = 0$ when $a \notin H$ or $\sum_{g \in aH} r_g = 1$ when $a \in H$, for each $a \in G$. Consequently, in the record of x there exists $0 \neq r_h \in R$ so that $h \in H$. Thus $x = h \in H$, as required. □

So, we come to the following affirmation.

Proposition 1.1. *Let $H \leq G_p$. Then $S(RG; H)$ a quasi-large subgroup of $S(RG)$ implies that H is a quasi-large subgroup of G_p.*

Proof. By virtue of the stated above criterion due to Benabdallah-Wilson we can write $S^{p^{m_i}}(RG)[p^i] \subset S(RG; H)$. Therefore, since $G^{p^{m_i}}[p^i] \subseteq S^{p^{m_i}}(RG)[p^i]$, we obtain $G^{p^{m_i}}[p^i] \subset S(RG; H) \cap G$. Henceforth, the Lemma works to infer that $G^{p^{m_i}}[p^i] \subset H$. Finally, again the aforementioned necessary and sufficient condition is applicable to get the claim. □

As a culmination we have the following criterion.

Theorem 1.1. *Let $H \leq G_p$ and $N(R) = 0$. Then $S(RG; H)$ is a quasi-large subgroup of $S(RG)$ if and only if H is a quasi-large subgroup of G_p.*

Proof. The necessity is precisely the foregoing Proposition. As for the sufficiency it is Theorem 1 from [2]. □

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