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MORE ON θ-COMPACT SPACES

MIGUEL CALDAS AND SAEID JAFARI

Abstract. In this paper, we present and study the notion of firm θ-continuity to investigate θ-compactness. We also present some properties of θ-compactness in terms of nets and ultranets. Moreover, we introduce and investigate some fundamental properties of θ-(m,n)-compact paces.

1. INTRODUCTION AND PRELIMINARIES

In 1943, Fomin [3] (see, also [4]) introduced the notion of θ-continuity. The notions of θ-open subsets, θ-closed subsets and θ-closure were introduced by Veličko [10] for the purpose of studying the important class of H-closed spaces in terms of arbitrary filterbases. Dickman and Porter [1], [2], Joseph [6] continued the work of Veličko. Recently Noiri and Jafari [8] have also obtained several new and interesting results related to these sets. In 1987, Noiri and Popa [3] introduced and investigated a new class of functions called quasi θ-continuous functions. The second author [5] further investigated quasi θ-continuity and also introduced the notion of θ-compactness by using θ-open sets introduced by Velicko [10]. Kupka [7] inspired by a number of characterizations of UC spaces (also called Atsuji spaces) [11] to characterize compact spaces. For this purpose, he asked the question that what kind of continuity should replace uniform to be sufficiently strong to characterize compact spaces. He was able to tackle this problem by introducing a new type of continuity called firm continuity by which he obtained several characterizations of compact spaces.

In section 2 of this paper, we continue the work of Kupka and obtain some characterizations of θ-compact spaces. In this relation we introduce and study the notion of firm θ-continuity which is natural for θ-compact spaces. In section 3, we give some more properties of θ-compactness in terms of nets and ultranets. In the last section, we introduce the new class of θ-(m,n)-compact spaces and investigate its fundamental properties.

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In what follows we denote the interior and the closure of a subset A of a topological space (X, τ) by $\mathrm{Int}(A)$ and $Cl(A)$, respectively. A point $x \in X$ is called a *θ-adherent point* of a subset A of X if $A \cap Cl(V) \neq \emptyset$ for every open set V containing x. The set of θ-adherent points of A is called the *θ-closure* of A which is denoted by $Cl_\theta(A)$. A subset A of X is called *θ-closed* if $A = Cl_\theta(A)$. The complement of a θ-closed set is called a *θ-open* set. The collection of all θ-closed (resp. θ-open) subsets of X will be denoted by $\theta C(X)$ (resp. $\theta O(X)$). We set

$$\theta C(X, x) = \{V \in \theta C(X) \colon x \in V\} \text{ for } x \in X.$$

We define similarly $\theta O(X, x)$. A topological space (X, τ) is said to be *θ-compact* [5] if every cover of X by θ-open sets has a finite subcover. A subset A of a topological space (X, τ) is said to be *θ-compact relative to τ* [5] if for any cover $\{V_i \mid i \in I\}$ of A by θ-open sets of (X, τ) there exists a finite subset I_0 of I such that $A \subset \bigcup\{V_i \mid i \in I_0\}$. A topological space (X, τ) is called θ-T_1 [5] if for every pair of distinct points x and y in X, there exists a θ-open set containing one of the points but not the other. It should be noted that θ-T_1 is equivalent with Hausdorff [5]. A function is said to be *quasi θ-continuous* [9] if every inverse image of each θ-open set in the codomain is θ-open in the domain.

2. CHARACTERIZATIONS OF θ-COMPACT SPACES

Definition 2.1. A function $f \colon X \to Y$, where X and Y are topological spaces, is said to have *property* Θ if for every θ-open cover ∇ of Y there exists a finite cover (the members of which need not be necessarily θ-open) $\{A_1, A_2, \ldots, A_n\}$ of X such that for each $i \in \{1, 2, \ldots, n\}$, there exists a set $U_i \in \nabla$ such that $f(A_i) \subset U_i$.

Lemma 2.1. *A topological space X is θ-compact if and only if for every topological space Y and every quasi θ-continuous function $f \colon X \to Y$, f has the property Θ.*

Proof. Let the topological space X be θ-compact and the function $f \colon X \to Y$ be quasi θ-continuous. Suppose that Ξ be a θ-open cover of Y. The set $f(X)$ is θ-compact relative to Y. This means that there exists a finite subfamily $\{U_1, U_2, \ldots, U_n\}$ of Ξ which cover $f(X)$. Then the sets

$$A_1 = f^{-1}(U_1),\ A_2 = f^{-1}(U_2), \ldots, A_n = f^{-1}(U_n)$$

form a cover of X such that $f(A_i) \subset U_i$ for each $i \in \{1, 2, \ldots, n\}$.

Conversely, suppose that X is a topological space such that for every topological space Y and every quasi θ-continuous function $f \colon X \to Y$, f has property Θ. It follows that the identity function $id_X \colon X \to X$ has also property Θ. Hence, for every θ-open cover Ξ of X, there exists a finite cover $A_1, A_2, \ldots, A_n$ of X such that for each $i \in \{1, 2, \ldots, n\}$ there exists a set $U_i \in \Xi$ such that $A_i = id_X(A_i) \subset U_i$. Then $U_1, U_2, \ldots, U_n$ are finite θ-subcover of Ξ. Since Ξ was an arbitrary θ-open cover of X, the space X is θ-compact. $\square$

Now we introduce the new class of firmly θ-continuous functions similar to the class of firmly continuous functions defined in [7].

Definition 2.2. A function $f\colon X \to Y$ is called *firmly θ-continuous* if for every θ-open cover ∇ of Y there exists a finite θ-open cover Ξ of X such that for every $U \in \Xi$, there exists a set $G \in \nabla$ such that $f(U) \subset G$.

Remark 2.1. It should be noted that if the topological space X is θ-compact and Y is an arbitrary topological space, then every quasi θ-continuous function $f\colon X \to Y$ is firmly θ-continuous.

Lemma 2.2. *Let X, Y, Z and W be topological spaces. Let $g\colon X \to Y$ and $h\colon Z \to W$ be quasi θ-continuous functions and let $f\colon Y \to Z$ be firmly θ-continuous. Then the functions $f \circ g\colon X \to Z$ and $h \circ f\colon Y \to W$ are firmly θ-continuous.*

Lemma 2.3. *Let X and Y be topological spaces. Suppose that $f\colon X \to Y$ is a quasi θ-continuous function which has the property Θ. Then f is firmly θ-continuous.*

Theorem 2.1. *The following statements are equivalent for a topological space (X, τ):*

(1) *X is θ-compact.*

(2) *The identity function $id_X\colon X \to X$ is firmly θ-continuous.*

(3) *Every quasi θ-continuous function from X to X is firmly θ-continuous.*

(4) *Every quasi θ-continuous function from X to a topological space Y is firmly θ-continuous.*

(5) *Every quasi θ-continuous function from X to a topological space Y has the property Θ.*

(6) *For each topological space Y and each quasi θ-continuous function $f\colon Y \to X$, f is firmly θ-continuous.*

Proof. (1)$\Rightarrow$ (2). Let X be θ-compact. The identity function $id_X\colon X \to X$ is quasi θ-continuous and by Remark 2.1 id_X is firmly θ-continuous.

(2)$\Rightarrow$ (3). Let $f\colon X \to X$ be any quasi θ-continuous function. By (2), the identity function $id_X\colon X \to X$ is firmly θ-continuous. Therefore by Lemma 2.2 $f = id_X \circ\colon X \to X$ is firmly θ-continuous.

(3)$\Rightarrow$ (4). Suppose that $f : X \to Y$ is any quasi θ-continuous function. The identity $id_X\colon X \to X$ is quasi θ-continuous and by (3) id_X is firmly θ-continuous. As a consequence of Lemma 2.2, we have that $f = f \circ id_X\colon X \to Y$ is firmly θ-continuous.

(4) $\Rightarrow$ (5). Obvious.

(5) $\Rightarrow$ (1). This is an immediate consequence of Lemma 2.1.

(6) $\Rightarrow$ (2). Suppose that $id_X\colon X \to X$ is the identity function. Then id_X is quasi θ-continuous and by (6) id_X is firmly θ-continuous.

(1) $\Rightarrow$ (6). Suppose that ∇ is a θ-open cover of X. Since X is θ-compact, then there is a finite θ-subcover $U_1, U_2, \ldots, U_n$ of ∇. Let $A_i = f^{-1}(U_i)$ for $i \in I$, where $I = \{1, 2, \ldots, n\}$. We have that $f(A_i) \subset U_i$ for every $i \in I$. Therefore f is firmly θ-continuous. $\square$

Theorem 2.2. *If $f\colon X \to Y$ is a firmly θ-continuous function, where X is a topological space and Y is a θ-T_1 topological space, then f is quasi θ-continuous.*

Proof. Let x be an arbitrary point of X and V be a θ-open set of Y containing $f(x)$. We define a θ-open cover Ξ of Y such that $\Xi = \{V, Y - f(x)\}$. Since f is firmly θ-continuous, it follows that there exists a finite θ-open cover $\{P_1, P_2, \ldots, P_n\}$ of X such that $f(P_i) \subset V$ or $f(P_i) \subset Y - f(x)$ for every $i \in \{1, 2, \ldots, n\}$. Let $x \in P_j$ for some index j. Since $f(P_j)$ contains $f(x)$, so it follows that $f(P_j) \subset V$. This shows that f is quasi θ-continuous. $\square$

3. PROPERTIES OF θ-COMPACT SPACES IN TERMS OF NETS AND ULTRANETS

Definition 3.1. Let (X, τ) be a topological space, $x \in X$ and $\{x_\ell, \ell \in L\}$ be a net of X. We say that a net $\{x_\ell, \ell \in L\}$ *θ-converges to x* if for each θ-open set U containing x, there exists an element $\ell_0 \in L$ such that $\ell \geq \ell_0$ implies $x_\ell \in U$.

Definition 3.2. Let (X, τ) be a topological space, $G = \{F_i\colon i \in I\}$ be a filterbase of X and $x \in X$. A filterbase G is said to *θ-converge to x* if there exists a member $F_i \in G$ such that $F_i \subseteq U$ for each θ-open set U containing x.

Proposition 3.1. *If $x \in U$ and $U \in \theta C(X, \tau)$, then there exists a net $\{x_i\}_{i \in I}$ that θ-converges to x and $x_i \in U$ for each $i \in I$.*

Proof. Suppose that $x \in U$ and $U \in \theta C(X, \tau)$ which means $U = Cl_\theta(U)$. This means that if $x \in N$ and $N \in \theta O(X, \tau)$ then $N \cap U \neq \emptyset$. It follows that there exists an element $x_N \in N \cap U$. This implies that $\{x_N\}_{N \in I}$ θ-converges to x. $\square$

Proposition 3.2. *Let $\{x_i\}_{i \in I}$ be a net in (X, τ) and $U \in \theta C(X, \tau)$, where $x_i \in U$ for each $i \in I$. If $\{x_i\}_{i \in I}$ θ-converges to x, then $x \in U$.*

Proof. Assume that $\{x_i\}_{i \in I}$ θ- converges to x and x does not belong to U. Then there exists a θ-open set N such that $x \in N$ and $N \cap U = \emptyset$. This means that there exists $i_0 \in I$ such that $x_i \in N$ for each $i \geq i_0$. Then x_i is not an element of U for each $i \geq i_0$. But this is a contradiction and hence the result. $\square$

Definition 3.3. A point y is a *θ-cluster point of $\{x_i\}_{i \in I}$* if for each $i_0 \in I$ and $U \in \theta O(X, \tau)$ such that $y \in U$, there exists an $i_1 \geq i_0$ such that $x_{i_1} \in U$.

Proposition 3.3. *Let $(\ell_i)_{i \in I}$ be an ultranet and y be a θ-cluster point of the net. Then the ultranet $(\ell_i)_{i \in I}$ θ-converges to y.*

Proof. Suppose that $(\ell_i)_{i\in I}$ is an ultranet in a topological space (X,τ) and y be a θ-cluster point of the net, $(\ell_i)_{i\in I}$. Suppose that, $(\ell_i)_{i\in I}$ doesn't θ-converge to y. This means that there exists $U \in \theta O(X,\tau)$ such that $y \in U$ and ℓ_i is not an element of U for any $i \in I$. So y is not a θ-cluster point of $(\ell_i)_{i\in I}$. □

Proposition 3.4. *Let $(\ell_i)_{i\in I}$ be a net in a topological space (X,τ). Then $y \in X$ is a θ-cluster point of $(\ell_i)_{i\in I}$, if and only if a subnet of $(\ell_i)_{i\in I}$ θ-converges to y.*

Proof. Let $(\ell_i)_{i\in I}$ have a subnet $(\ell_{k_j})_{j\in J}$ that θ-converges to y and J be a directed set. Now suppose that $y \in X$ is not a θ-cluster point of $(\ell_i)_{i\in I}$. This means that there exists $U \in \theta O(X,\tau)$ and $i_0 \in I$ such that, s_{i_1} is not an element of U for every $i_1 \geq i_0$. Then $(\ell_{k_j})_{j\in J}$ doesn't θ-converge to y.

Conversely assume that y is a θ-cluster point of $(\ell_i)_{i\in I}$.

$J = \{(i,U)\colon i \in I, y \in U, U \in \theta O(X,\tau) \text{ and } \ell_i \in U\}$ is a partially ordered set where $(i,U) \leq (i_1,V)$, if $i \leq i_1$ and $V \subset U$.

(i) $(i,U) \leq (i,U)$ for every $(i,U) \in J$. Because, $i \leq i$ and $U \subset U$ for every $i \in I$ and $U \in \theta O(X,\tau)$.

(ii) Let $(i,U) \leq (i_1,V)$ and $(i_1,V) \leq (i,U)$. Then, $i \leq i_1$, $V \subset U$ and $i_1 \leq i$, $U \subset V$. This follows that $i = i_1$, $V = U$. Then, $(i_1,V) = (i,U)$.

(iii) Let $(i,U),(i_1,V)$ and $(i_2,W) \in J$ such that $(i,U) \leq (i_1,V)$ and $(i_1,V) \leq (i_2,W)$. Since I is a directed set, $i \leq i_2$ where $i \leq i_1$ and $i_1 \leq i_2$. Also, we know that $W \subset U$ where $V \subset U$ and $W \subset V$. Then, $(i,U) \leq (i_2,W)$ where $i \leq i_2$ and $W \subset U$. Consequently, J is a partially ordered set.

Now let $(i,U),(i_1,V) \in J$. Then $U \cap V \in \theta O(X,\tau)$. We know that $U \cap V \subset U$ and $U \cap V \subset V$ and $y \in U \cap V$. Since y is a θ-cluster point of $(\ell_i)_{i\in I}$, there exists $i_2 \in I$ such that $i \leq i_2$, $i_1 \leq i_2$ and $s_{i_2} \in U \cap V$. Then $(i_1,V) \leq (i_2,U\cap V)$ and $(i,U) \leq (i_2,U\cap V)$. This means that J is a directed set. Define $k\colon J \to I$ by $k(i,A) = i$.

(a) $(i,U) \leq (i_1,V)$ means that $i \leq i_1$. Then $k(i,U) \leq k(i_1,V)$.

(b) Let $i,i_1 \in I$ and $U \in \theta O(X,\tau)$ which contains y. Then there exists $i_2 \in I$ such that $i \leq i_2$, $i_1 \leq i_2$ and $\ell_{i_2} \in U$. This means that $(i_2,U) \in J$, $i \leq k(i_2,U)$ and $i_1 \leq k(i_1,U)$. This follows that $\{\ell_{k(i,U)}\}_{(i,U)\in J}$ is a subnet of $\{\ell_i\}_{i\in I}$.

Consider the set $U \in \theta O(X,\tau)$ which contains y. There exists $i_0 \in I$ such that $\ell_{i_0} \in U$. Then $(i_0,U) \in J$. For every $(i,V) \in J$ that $(i_0,U) \leq (i,V)$, $V \subset U$ and $\ell_i \in V$. This follows that $\ell_{k(i,V)} \in U$ for every $(i_0,U) \leq (i,V)$. So the subnet, $\{\ell_{k(i,U)}\}_{(i,U)\in J}$, θ-converges to y. □

Proposition 3.5. *Let (X,τ) be topological space. Then the following statements are equivalent:*

(i) *(X,τ) is θ-compact.*

(ii) *For any family* Ψ *of* θ*-closed subsets of* X *such that* $\bigcap_{K\in\Psi} K = \emptyset$, *there exists a finite subfamily* $\Phi \subset \Psi$ *such that* $\bigcap_{L\in\Phi} L = \emptyset$.

(iii) $\bigcap_{K\in\Psi} K \neq \emptyset$ *for any family* Ψ *of* θ*-closed subsets of* X *such that* $\bigcap_{L\in\Phi} L \neq \emptyset$ *where* $\Phi \subset \Psi$ *is a finite subfamily.*

Proof. (i) $\Rightarrow$ (ii). Let (X,τ) be θ-compact and Ψ be a family of θ-closed subsets such that $\bigcap_{K\in\Psi} K = \emptyset$. Then $\left[\bigcap_{K\in\Psi} K\right]^c = [\emptyset]^c$. This means that $\bigcup_{K\in\Psi} K^c = X$. There exists a finite subfamily $\Phi \subset \Psi$ such that $\bigcup_{L\in\Phi} L^c = X$ where $\bigcap_{L\in\Phi} L = \emptyset$.

(ii) $\Rightarrow$ (iii). Let Ψ be a family of θ-closed subsets of X. From the assumption if $\bigcap_{K\in\Psi} K = \emptyset$, then there exists a finite subfamily $\Phi \subset \Psi$ such that $\bigcap_{L\in\Phi} L = \emptyset$. This means that if Ψ doesn't have any finite subfamily Φ such that $\bigcap_{L\in\Phi} L = \emptyset$, then $\bigcap_{K\in\Psi} K \neq \emptyset$.

(iii) $\Rightarrow$ (ii). Let Ψ be a family of θ-closed subsets of X. From the assumption, if $\bigcap_{L\in\Phi} L \neq \emptyset$ for any subfamily $\Phi \subset \Psi$ then $\bigcap_{K\in\Psi} K \neq \emptyset$. This means that, if $\bigcap_{K\in\Psi} K = \emptyset$ then there exists at least one subfamily $\Phi \subset \Psi$ such that $\bigcap_{L\in\Phi} L = \emptyset$.

(ii) $\Rightarrow$ (i). Let $\{U_i\}_{i\in I}$ be a θ-open cover of X. Then, $\bigcup_{i\in I} U_i = X$. This means that $\bigcap_{i\in I} U_i^c = \emptyset$ and $U_i^c \in \theta C(X,\tau)$ for each $i \in I$. It follows from the assumption that there exists a finite subfamily $J \subset I$ such that $\bigcap_{j\in J} U_j^c = \emptyset$. So $\bigcup_{j\in J} U_j = X$. Therefore (X,τ) is θ-compact. ☐

Proposition 3.6. *A topological space* (X,τ) *is* θ*-compact if and only if every net has at least one* θ*-cluster point in the topological space.*

Proof. Let (X,τ) be θ-compact and $\{x_i\}_{i\in I}$ be any net in this space. Let's consider a family $Cl_\theta(B_j)$ of subsets where $B_j = \{x_i \mid j \leq i\}$. Then, $Cl_\theta(B_j) \in \theta C(X,\tau)$ for any $j \in I$ and the intersection of finitely many of $Cl_\theta(B_j)$ is nonempty. It follows from Proposition 3.5 that $\bigcap_{j\in I} Cl_\theta(B_j) \neq \emptyset$ for (X,τ) is θ-compact. Let $y \in \bigcap_{j\in I} Cl_\theta(B_j)$. Then $y \in Cl_\theta(B_j)$ for any $j \in I$. Consider $y \in U$, $U \in \theta O(X,\tau)$ and $r \in I$. Then $U \cap B_r \neq \emptyset$. So $U \cap B_k \neq \emptyset$ for any $k \in I$ such that $k \geq r$. Consequently y is a θ-cluster point of $\{x_i\}_{i\in I}$.

Now suppose that every net in (X,τ) has at least one θ-cluster point. Let $\{F_i\}_{i\in I}$ be a family of θ-closed sets where intersection of finitely many of $F_i's$ is nonempty.

Consider the set $J = \left\{ \bigcap\limits_{j=1}^{n} G_{i_j} \mid \{G_{i_j}\}_{j=1}^{n} \subset \{F_i\}_{i\in I} \right\}$ and the relation "$\leq$", where $A \leq B$ whenever $B \subset A$ and $A, B \in J$.

(i) $A \subset A$ for every set $A \in J$. This means that $A \leq A$ for every set $A \in J$.

(ii) We know that if $A \supset B$ and $B \supset A$ then $A = B$. So $A \leq B$ and $B \leq A$ then $A = B$.

(iii) We know that if $C \supset B$ and $B \supset A$ then $C \supset A$. So, if $C \leq B$ and $B \leq A$ then $C \leq A$. This means that $(J, \leq)$ is a directed set and partially ordered.

Let us consider the function $\ell \colon J \to X$ such that $\ell(A) \in A$ for every $A \in J$. Then $\{\ell_A\}_{A\in J}$ is a net in X and by the assumption has a θ-cluster point. Let y be the θ-cluster point of $\{\ell_A\}_{A\in J}$. We know that if $A \in J$ and $F_k \leq A$, then $A \subset F_k$ where $F_k \in \{F_i\}_{i\in I}$. So $\ell_B \in F_k$ whenever $A \leq B$. Then, $\{\ell_A\}_{A\in J}$ is residually in F_k. By Proposition 3.4, since y is a θ-cluster point of $\{\ell_A\}_{A\in J}$, a subnet of $\{\ell_A\}_{A\in J}$ θ-converges to y. Since $\{\ell_A\}_{A\in J}$ is residually in F_k for each k, such a subnet would be residually in F_k for each k. By Proposition 3.2, $y \in F_k$ for each k. So $\bigcap\limits_{i\in I} F_i \neq \emptyset$. By Proposition 3.5, (X, τ) is θ-compact. □

Proposition 3.7. *A topological space (X, τ) is θ-compact if and only if every ultranet in it is θ-convergent.*

Proof. Suppose (X, τ) is θ-compact and $\{\ell_i\}_{i\in I}$ is an ultranet in (X, τ). By Proposition 3.6, $\{\ell_i\}_{i\in I}$ has at least one θ-cluster point. From Proposition 3.3, $\{\ell_i\}_{i\in I}$ θ-converges to its θ-cluster point. Hence, $\{\ell_i\}_{i\in I}$ is θ-convergent.

Conversely, assume that every ultra net in (X, τ) is θ-convergent. Let $\{\ell_i\}_{i\in I}$ be a net in (X, τ). Since every net has a subnet which is an ultranet, so there exists a subnet of $\{\ell_i\}_{i\in I}$ which is an ultranet. This ultranet θ-converges to a point which is θ-cluster point of $\{\ell_i\}_{i\in I}$. □

4. θ-(m, n)-COMPACT SPACES

We begin with the following notions which will be used in the sequel.

Definition 4.1. A space (X, τ) is said to be θ-(m, n)-*compact* if from every θ-open covering $\{U_i \mid i \in I\}$ of X whose cardinality I, denoted by Card I, is at most n, one can select a subcovering $\{U_{i_j} \mid j \in J\}$ of X whose Card J is at most m.

Definition 4.2. A subset A of a space (X, τ) is said to be a θ-(m, n)-*compact subspace* if the subspace A is θ-(m, n)-compact.

Definition 4.3. A space (X, τ) is said to be *completely* θ-(m, n)-*compact* if every subspace of X is θ-(m, n)-compact.

Remark 4.1. It should be noted that a θ-$(1,n)$-compact space is a θ-n-compact space and θ-$(1,\infty)$-compact space is the usual θ-compact space. A θ - $(1,\omega)$ - compactness is θ-compactness in the Fréchet sense and a θ-(ω,∞)-compact space is a θ-Lindelöf space.

Definition 4.4. A family $\{U_i \mid i \in I\}$ of subsets of a set X is said *to have the m-intersection property* if every subfamily of cardinality at most m has a non-void intersection.

Theorem 4.1. *A space (X,τ) is θ-(m,n)-compact if and only if every family $\{P_i\}$ of θ-closed sets $P_i \subseteq X$ having the m-intersection property also has the n-intersection property.*

Proof. The proof is a consequence of the following equivalent statements:

(1) X is θ-(m,n)-compact.

(2) If $\{U_i \mid i \in I\}$ is a θ-open cover of X such that Card $I \leq n$, then there is a subcover $\{U_{i_j} : j \in J\}$ of X such that Card $J \leq m$.

(3) If $\{U_i \mid i \in I\}$ is a family of θ-open sets such that Card $I \leq n$ and every subfamily $\{U_{i_j}\}$ of Card $J \leq m$ has the property $X - \Big(\bigcup\limits_i U_{i_j}\Big) \neq \emptyset$, then $X - \Big(\bigcup\limits_{i\in I} U_i\Big) \neq \emptyset$.

(4) If $\{U_i \mid i \in I\}$ is a family of θ-open sets such that $X - \Big(\bigcup\limits_{j\in J} U_{i_j}\Big) \neq \emptyset$ whenever Card $J \leq m$, then $X - \Big(\bigcup\limits_{j\in J} U_{i_j}\Big) \neq \emptyset$ whenever Card $J \leq n$.

(5) If $\{P_i \mid i \in I\}$ is a family of θ-closed sets having the m-intersection property then $\{P_i\}$ has also the n-intersection property. □

Theorem 4.2. *If a space X is θ-(m,n)-compact and if Y is a θ-closed subset of X, then Y is a θ-(m,n)-compact subspace of X.*

Proof. Suppose that $\{U_i \mid i \in I\}$ is a θ-open cover of Y such that Card $I \leq n$. By adding $U_j = X - Y$, we obtain a θ-open cover of X with cardinality at most n. By eliminating U_j, we have a subcover of $\{U_i\}$ whose cardinality is at most m. □

Theorem 4.3. *If X is a space such that every θ-open subset of X is a θ-(m,n)-compact subspace of X, then X is completely θ-(m,n)-compact.*

Proof. Let $Y \subset X$ and $\{U_i \mid i \in I\}$ be a θ-open cover of Y such that Card $I \leq n$. Then the family $\{U_i \mid i \in I\}$ is a θ-open cover of the θ-open set $\bigcup\limits_i U_i$. Then there is a subfamily $\{U_{i_j} \mid j \in J\}$ of Card $J \leq m$ which covers $\bigcup\limits_i U_i$. This subfamily also covers the set Y and therefore Y is θ-(m,n)-compact. □

Theorem 4.4. *Let X be a topological space and $\{Y_k \mid k \in K\}$ be a family of subsets of X. If every Y_k is θ-(m,n)-compact for some $m \geq$ Card K, then $U_{k \in K} Y_k$ is a θ-(m,n)-compact subspace of X.*

Proof. If $\{U_i \mid i \in I\}$ is a θ-open cover of $Y = \bigcup\limits_k Y_k$, then it is a θ-open cover of Y_k for every $k \in K$. If Card $I \leq n$, then $\{U_i\}$ contains a subfamily $\{U_{i_{j_k}} \mid j_k \in J_k\}$ for which $CarJ_k \leq m$ and is a covering of Y_k. The union of these families is a θ-open subfamily of $\{U_i\}$ which covers Y and its cardinality is at most m. □

Definition 4.5. A point $x \in X$ is said to be an *m-θ-accumulation point of a set S in X* if for every θ-open set U_x containing x, we have Card $(U_x \cap S) > m$.

It should be noted that if $m = 0, 1$ or ω, then the relation Card $(U_x \cap S) > m$ means that $U_x \cap S \neq \emptyset$, not finite or not countable.

Theorem 4.5. *Let X be a topological space and $S \subset X$ and Card $S > m$. If X is θ-(m,n)-compact for some $n > m$, then S has a θ-accumulation point in X. If X is θ-(m,∞)-compact, then S has an m-θ-accumulation point in X.*

Proof. Let $S \subset X$ and S be of cardinality at most n which has no θ-accumulation points in X. Then for each $x \in X$, there is an open set U_x such that at most one point of S belongs to U_x. Suppose U is the union of all sets U_x which contain no points of S. Let U_s denote the union of all sets U_x which contain the point $s \in S$. Then U and U_s are θ-open sets. Therefore $\{U, U_s\}$ is a θ-open cover of X of cardinality at most n. If X is θ-(m,n)-compact, then this cover contains a subcover of cardinality at most n. If X is θ-(m,n)-compact, then this cover contains a subcover of cardinality at most m. But this subcover must contain every U_s since $s \in S$ is covered only by U_s. Hence Card $S \leq m$. If the cardinality of a set S is greater than m, then S has at least one θ-accumulation point in X. The two other cases can be proved similarly with a little modification. □

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Universidade Federal Fluminense
Departamento de Matemática Aplicada
Str Rua Mário Santos Braga, s/n, 24020-140*, Niterói, RJ-Brasil*
E-mail address: `gmamccs@vm.uff.br`

College of Vestsjaelland South
Str Herrestraede 11*,* 4200*, Slagelse, Denmark*
E-mail address: `jafari@stofanet.dk`