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ON SEMINORMED SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

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ABSTRACT. In this paper we define the sequence space $\ell(\mathbf{M}, p, q, u)$ on a seminormed complex linear space, by using a sequence of Orlicz functions. We study its some algebraic and topological properties. We give also some inclusion relations. This study generalized some results of Bektaş and Altın [1].

1. INTRODUCTION

Let ω be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$ where $k \in \mathbb{N}$, the set of positive integers.

An Orlicz function is a function $M\colon [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

If M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$.

Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \quad \text{for some } \rho > 0 \right\}.$$

The space ℓ_M becomes a Banach space, with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

which is called an *Orlicz space*. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

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Let X be a complex linear space with zero element θ and $X = (X, q)$ be a seminormed space with the seminorm q. By $S(X)$ we denote the linear space of all sequences $x = (x_k)$ with $(x_k) \in X$ and the usual coordinatewise operations: $\alpha x = (\alpha x_k)$ and $x + y = (x_k + y_k)$, for each $\alpha \in \mathbb{C}$ where $\mathbb{C}$ denotes the set of complex numbers. If $\lambda = (\lambda_k)$ is a scalar sequence and $x \in S(X)$ then we shall write $\lambda x = (\lambda_k x_k)$.

Let U be the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ and complex for all $k = 1, 2, \ldots$.

Let $\mathbf{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a sequence of positive real numbers and X be a seminormed space with seminorm q. Given $u \in U$. Then we define

$$\ell(\mathbf{M}, p, q, u) = \left\{ x \in S(X) : \sum_{k=1}^{\infty} u_k \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} < \infty, \ \rho > 0 \right\}.$$

We get the following sequence spaces from $\ell(\mathbf{M}, p, q, u)$ on giving particular values to p and u. Taking $p_k = 1$ for all $k \in \mathbb{N}$ we have

$$\ell(\mathbf{M}, q, u) = \left\{ x \in S(X) : \sum_{k=1}^{\infty} u_k \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right] < \infty, \ \rho > 0 \right\}.$$

If we take $u_k = 1$, then we have

$$\ell(\mathbf{M}, p, q) = \left\{ x \in S(X) : \sum_{k=1}^{\infty} \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} < \infty, \ \rho > 0 \right\}.$$

If we take $p_k = 1$ and $u_k = 1$ for all $k \in \mathbb{N}$, then we have

$$\ell(\mathbf{M}, q) = \left\{ x \in S(X) : \sum_{k=1}^{\infty} \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right] < \infty, \ \rho > 0 \right\}.$$

In addition to the above sequence spaces, we have $\ell(\mathbf{M}, p, q, u) = \ell_M(p)$ due to Parashar and Choudhary [5], on taking $u_k = 1$ for all $k \in \mathbb{N}$, $q(x) = |x|$, $(M_k) = M$ for all $k \in \mathbb{N}$ and $X = \mathbb{C}$.

A sequence space E is said *to be solid (or normal)* if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$ [2].

A sequence space E is said to be monotone if it contains the canonical preimages of all its step-space.

It is well known that a sequence space E is normal implies that E is monotone.

The following inequality and $p = (p_k)$ sequence will be used frequently throughout this paper.

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}, \tag{1.1}$$

where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup_k p_k = G$, $D = \max(1, 2^{G-1})$ [4].

2. MAIN RESULTS

In this section we will prove the results of this article involving the sequence space $\ell(\mathbf{M}, p, q, u)$.

Theorem 2.1. *The sequence space $\ell(\mathbf{M}, p, q, u)$ is a linear space over the field $\mathbb{C}$ complex numbers.*

Proof. Let $x, y \in \ell(\mathbf{M}, p, q, u)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist some positive numbers ρ_1 and ρ_2 such that

$$\sum_{k=1}^{\infty} u_k \left[M_k \left(q \left(\frac{x_k}{\rho_1} \right) \right) \right]^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty} u_k \left[M_k \left(q \left(\frac{y_k}{\rho_2} \right) \right) \right]^{p_k} < \infty.$$

Define $\rho_3 = \max\big(2|\alpha|\rho_1, 2|\beta|\rho_2\big)$. Since M_k are non-decreasing convex functions and q is a seminorm, we have from (1.1)

$$\sum_{k=1}^{\infty} u_k \left[M_k \left(q \left(\frac{\alpha x_k + \beta y_k}{\rho_3} \right) \right) \right]^{p_k} \leq D \sum_{k=1}^{\infty} u_k \left[M_k \left(q \left(\frac{x_k}{\rho_1} \right) \right) \right]^{p_k}$$
$$+ D \sum_{k=1}^{\infty} u_k \left[M_k \left(q \left(\frac{y_k}{\rho_2} \right) \right) \right]^{p_k} < \infty.$$

□

Theorem 2.2. *The sequence space $\ell(\mathbf{M}, p, q, u)$ is paranormed (not necessarily totally paranormed) space, paranormed by*

$$g(x) = \inf \left\{ \rho^{p_n/H} : \sup_k \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right] \leq 1, \quad n \in \mathbb{N}, \ \rho > 0 \right\}$$

where $H = \max(1, \sup_k p_k)$.

Proof. Clearly $g(x) = g(-x)$. Let $x \in \ell(\mathbf{M}, p, q, u)$. Then there exist $\rho_1 > 0$, $\rho_2 > 0$ such that

$$\sup_k \left[M_k \left(q \left(\frac{x_k}{\rho_1} \right) \right) \right] \leq 1$$

and

$$\sup_k \left[M_k \left(q \left(\frac{y_k}{\rho_2} \right) \right) \right] \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we get the following inequality from the triangle inequality

$$\sup_k \left[M_k \left(q \left(\frac{x_k + y_k}{\rho} \right) \right) \right] \leq \frac{\rho_1}{\rho} \sup_k \left[M_k \left(q \left(\frac{x_k}{\rho_1} \right) \right) \right] + \frac{\rho_2}{\rho} \sup_k \left[M_k \left(q \left(\frac{y_k}{\rho_2} \right) \right) \right] \leq 1.$$

Hence

$$\begin{aligned} g(x+y) &= \inf \left\{ \rho^{p_n/H} : \sup_k \left[M_k \left(q \left(\frac{x_k + y_k}{\rho} \right) \right) \right] \leq 1,\ n \in \mathbb{N} \right\} \\ &\leq \inf \left\{ (\rho_1 + \rho_2)^{p_n/H} : \sup_k \left[M_k \left(q \left(\frac{x_k}{\rho_1} \right) \right) \right] \leq 1, \sup_k \left[M_k \left(q \left(\frac{y_k}{\rho_2} \right) \right) \right] \leq 1 \right\} \\ &\leq g(x) + g(y). \end{aligned}$$

Since $q(\theta) = 0$ and $M_k(0) = 0$ for all $k \in \mathbb{N}$, we get $\inf\{\rho^{p_n/H}\} = 0$ for $x = \theta$.

Finally, we prove that scalar multiplication is continuous. Let λ be any number. From definition, we have

$$\begin{aligned} g(\lambda x) &= \inf \left\{ \rho^{p_n/H} :\ \sup_k \left[M_k \left(q \left(\frac{\lambda x_k}{\rho} \right) \right) \right] \leq 1,\ n \in \mathbb{N} \right\} \\ &= \inf \left\{ (|\lambda|\, r)^{p_n/H} :\ \sup_k \left[M_k \left(q \left(\frac{x_k}{r} \right) \right) \right] \leq 1,\ n \in \mathbb{N} \right\} \end{aligned}$$

where $r = \dfrac{\rho}{|\lambda|}$.

Now it can be easily verified that $\lambda \to 0$ and x fixed implies $g(\lambda x) \to 0$; λ fixed and $x \to \theta$ implies $g(\lambda x) \to 0$ in $\ell(\mathbf{M}, p, q, u)$; $\lambda \to 0$ and $x \to \theta$ implies $g(\lambda x) \to 0$ in $\ell(\mathbf{M}, p, q, u)$. □

Theorem 2.3. *Let $\mathbf{M} = (M_k)$ and $\mathbf{T} = (T_k)$ be any two sequences of Orlicz functions. Then we have:*

(i) $\ell(\mathbf{M}, p, q, u) \cap \ell(\mathbf{T}, p, q, u) \subseteq \ell(\mathbf{M} + \mathbf{T}, p, q, u)$.

(ii) *In case of u_k and v_k are sequences of real numbers, if $v_k \leq u_k$, then $\ell(\mathbf{M}, p, q, u) \subseteq \ell(\mathbf{M}, p, q, v)$.*

Proof. (i) From (1) we have

$$\begin{aligned} u_k \left[(M_k + T_k) \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} &= u_k \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) + T_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} \\ &\leq D\ u_k \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} + D\ u_k \left[T_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{p_k}. \end{aligned}$$

Let $x \in \ell(\mathbf{M}, p, q, u) \cap \ell(\mathbf{T}, p, q, u)$; when adding the above inequality from $k = 1$ to ∞ , we get $x \in \ell(\mathbf{M} + \mathbf{T}, p, q, u)$.

Proof of (ii) is trivial. □

Theorem 2.4. *Let* $0 < p_k \le t_k < \infty$ *for each* $k \in \mathbb{N}$. *Then*

$$\ell(\mathbf{M}, p, q, u) \subseteq \ell(\mathbf{M}, t, q, u).$$

Proof. Let $x \in \ell(\mathbf{M}, p, q, u)$. Then there exists some $\rho > 0$ such that

$$\sum_{k=1}^{\infty} u_k \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} < \infty.$$

This implies that $M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \le 1$ for sufficiently large values of k, say $k \ge k_0$ for some fixed $k_0 \in \mathbb{N}$. Since $p_k \le t_k$ for each $k \in \mathbb{N}$, we get

$$\left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{t_k} \le \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{p_k}$$

for all $k \ge k_0$ and therefore

$$\sum_{k \ge k_0} u_k \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{t_k} \le \sum_{k \ge k_0} u_k \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} < \infty.$$

Hence $x \in \ell(\mathbf{M}, t, q, u)$. □

Theorem 2.5. *In case of* u_k *is a sequence of real numbers:*

(i) *If* $0 < u_k \le 1$ *for all* $k \in \mathbb{N}$, *then* $\ell(\mathbf{M}, q) \subseteq \ell(\mathbf{M}, q, u)$.

(ii) *If* $u_k \ge 1$ *for all* $k \in \mathbb{N}$, *then* $\ell(\mathbf{M}, q, u) \subseteq \ell(\mathbf{M}, q)$.

Proof is trivial.

Theorem 2.6. *The sequence space* $\ell(\mathbf{M}, p, q, u)$ *is solid.*

Proof. Let $(x_k) \in \ell(\mathbf{M}, p, q, u)$, i.e.

$$\sum_{k=1}^{\infty} u_k \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} < \infty.$$

Let (α_k) be sequence of scalars such that $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$. Then the result follows from the following inequality

$$\sum_{k=1}^{\infty} u_k \left[M_k \left(q \left(\frac{\alpha_k x_k}{\rho} \right) \right) \right]^{p_k} \le \sum_{k=1}^{\infty} u_k \left[M_k \left(q \left(\frac{x_k}{\rho} \right) \right) \right]^{p_k}.$$

□

Corollary 2.1. *The sequence space* $\ell(\mathbf{M}, p, q, u)$ *is monotone.*

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