

SOME FIXED POINT THEOREMS ON DUALISTIC PARTIAL METRIC SPACES

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ABSTRACT. In this paper, we give an order relation on dualistic partial metric space. Later, using this relation, some fixed point theorems for single and multi valued mappings on ordered dualistic partial metric space are proved.

1. INTRODUCTION

Recently, Matthews [5] introduced the notion of a partial metric space and obtained a Banach fixed point theorem for complete partial metric spaces. After then O'Neill defined the concept of dualistic partial metric, which is general then partial metric. In [7], Oltra and Valero gave a Banach fixed point theorem on complete dualistic partial metric spaces. Also in [7], it was showed that the contractive condition in Banach fixed point theorem on complete dualistic partial metric spaces can not be replaced by the contractive condition of Banach fixed point theorem for complete partial metric spaces. Later, Valero [8] has generalized the main theorem of [7] using nonlinear contractive condition instead of Banach contractive condition. As it can be understand above, fixed point theory on dualistic partial metric or partial metric spaces have been done for contractive or contractive type mappings. In this paper, we introduce a partial ordering on dualistic partial metric spaces utilizing partial metric function and use the same to prove a fixed point theorem for single valued nondecreasing mappings on ordered dualistic partial metric spaces. Similar results are also proved for multi valued mappings.

2. PRELIMINARIES

In this section, as in [7] and [8], we give definitions and some properties of partial metric and dualistic partial metric spaces. Throughout this paper the letters \mathbb{R} , \mathbb{R}^+ and \mathbb{N} will denote the set of real numbers, the set of nonnegative real numbers and the set of natural numbers, respectively.

The notion of a partial metric space was introduced by Matthews in [5] as a part of the study of denotational semantics of dataflow networks. In particular,

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he established the precise relationship between partial metric spaces and so called weightable quasi-metric spaces, and proved a partial metric generalization of Banach's contraction mapping theorem.

Let us recall that a partial metric on a nonempty set X is a function $p: X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$:

- (p₁) $x = y \iff p(x, x) = p(x, y) = p(y, y)$;
- (p₂) $p(x, x) \leq p(x, y)$;
- (p₃) $p(x, y) = p(y, x)$;
- (p₄) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X .

In [6] O'Neill proposed one significant change to Matthews' definition of partial metrics, and that was to extend their range from \mathbb{R}^+ to \mathbb{R} .

In the following, partial metrics in the O'Neill sense will be called dualistic partial metrics and a pair (X, p) such that X is nonempty set and p is dualistic partial metric on X will be called a dualistic partial metric space.

In this way O'Neill developed several connections between partial metrics and the topological aspects of domain theory. Moreover, the pair (\mathbb{R}, p) , where

$$p(x, y) = x \vee y \text{ for all } x, y \in \mathbb{R},$$

provides a paradigmatic example of a dualistic partial metric space that is not a partial metric space. Other examples of dualistic partial metric or partial metric spaces which are interesting from a computational point of view may be found in [1], [5].

Each dualistic partial metric p on X generates a T_0 topology $\tau(p)$ on X which has as a base the family open p -balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\} \text{ for all } x \in X \text{ and } \varepsilon > 0.$$

From this fact it immediately follows that a sequence $\{x_n\}$ in a dualistic partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n)$.

According to [6], a sequence $\{x_n\}$ in a dualistic partial metric space (X, p) is called a Cauchy sequence if there exists $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$.

A dualistic partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to $\tau(p)$, to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$.

Now we establish some known correspondences between dualistic partial metric and quasi-metric spaces. See for details about quasi-metric spaces in [3] and [4]. By a quasi-metric on a set X , we mean a nonnegative real valued function d on $X \times X$ such that for all $x, y, z \in X$:

- (d₁) $d(x, y) = d(y, x) = 0 \iff x = y$;
- (d₂) $d(x, y) \leq d(x, z) + d(z, y)$.

A quasi-metric space is a pair (X, d) such that X is a nonempty set and d is a quasi-metric on X .

Each quasi-metric d on X generates a T_0 topology $\tau(d)$ on X which has as a base the family open d -balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\} \text{ for all } x \in X \text{ and } \varepsilon > 0.$$

If d is a quasi-metric on X , then the function d^s defined on $X \times X$ by

$$d^s(x, y) = \max\{d(x, y), d(y, x)\},$$

is a metric on X .

The proof of the following auxiliary results can be found in [7].

Lemma 2.1. *If (X, p) is a dualistic partial metric space, then the function $d_p : X \times X \rightarrow \mathbb{R}^+$ defined by $d_p(x, y) = p(x, y) - p(x, x)$ is a quasi-metric on X such that $\tau(p) = \tau(d_p)$.*

Lemma 2.2. *Let (X, p) be a dualistic partial metric spaces. Then the following assertions are equivalent:*

- (1) (X, p) is complete.
- (2) The induced metric space $(X, (d_p)^s)$ is complete.

Furthermore $\lim_{n \rightarrow \infty} (d_p)^s(a, x_n) = 0$ if and only if

$$p(a, a) = \lim_{n \rightarrow \infty} p(a, x_n) = \lim_{n, m \rightarrow \infty} p(x_n, x_m).$$

In [8], it was proved the following fixed point theorem which generalize the Banach fixed point theorem on dualistic partial metric spaces given in [7].

Theorem 2.1. *Let f be a mapping of a complete dualistic partial metric space (X, p) into itself such that*

$$|p(fx, fy)| \leq \Phi(|p(x, y)|) \text{ for all } x, y \in X,$$

where $\Phi : [0, \infty) \rightarrow [0, \infty)$ is any monotone non-decreasing function with $\lim_{n \rightarrow \infty} \Phi^n(t) = 0$ for each fixed $t > 0$. Then f has a unique fixed point.

3. MAIN RESULTS

We begin by proving the following lemma.

Lemma 3.1. *Let (X, p) be a dualistic partial metric space and $\phi : X \rightarrow \mathbb{R}$. Define the relation " \preceq " on X as follows;*

$$x \preceq y \iff 2p(x, y) - p(x, x) - p(y, y) \leq \phi(x) - \phi(y).$$

Then " \preceq " is a (partial) order on X , named the partial order induced by ϕ .

Proof. For all $x \in X$, it is obvious that $x \preceq x$, then “ \preceq ” is reflexive. Again for $x, y \in X$, let $x \preceq y$ and $y \preceq x$. Then,

$$2p(x, y) - p(x, x) - p(y, y) \leq \phi(x) - \phi(y) \quad (3.1)$$

and

$$2p(y, x) - p(y, y) - p(x, x) \leq \phi(y) - \phi(x). \quad (3.2)$$

Therefore, from (3.1), (3.2) and (p₃), we have

$$2p(x, y) - p(x, x) - p(y, y) \leq 0.$$

Now from Lemma 2.1, we have

$$d_p(x, y) + d_p(y, x) \leq 0.$$

Since d_p is non-negative value, we have $d_p(x, y) = 0 = d_p(y, x)$, this shows that $x = y$, that is, “ \preceq ” is antisymmetric. Now for $x, y, z \in X$, let $x \preceq y$ and $y \preceq z$, then,

$$2p(x, y) - p(x, x) - p(y, y) \leq \phi(x) - \phi(y)$$

and

$$2p(y, z) - p(y, y) - p(z, z) \leq \phi(y) - \phi(z).$$

On the other hand

$$\begin{aligned} 2p(x, z) - p(x, x) - p(z, z) &= p(x, z) + p(x, z) - p(x, x) - p(z, z) \\ &\leq p(x, y) + p(y, z) - p(y, y) - p(x, x) \\ &\quad + p(x, y) + p(y, z) - p(y, y) - p(z, z) \\ &= 2p(x, y) - p(x, x) - p(y, y) + 2p(y, z) - p(y, y) - p(z, z) \\ &\leq \phi(x) - \phi(z) \end{aligned}$$

that is, $x \preceq z$. Therefore “ \preceq ” is transitive. \square

Now we give some examples.

Example 3.1. Let $X = \mathbb{R}$ and $p(x, y) = x \vee y$. Consider the function ϕ defined by $\phi(x) = -x$. Now for $x, y \in X$, then

$$\begin{aligned} x \preceq y &\iff 2p(x, y) - p(x, x) - p(y, y) \leq \phi(x) - \phi(y) \\ &\iff 2(x \vee y) - x - y \leq -x + y \\ &\iff x \vee y \leq y \end{aligned}$$

According to this, the partial \preceq order is usual order on X .

Example 3.2. Let $X = \{1, 2, \dots\}$ and p as in Example 3.1. Let $\phi(x) = 1 - \frac{6}{x}$. Now for $x, y \in X$, then

$$\begin{aligned} x \preceq y &\iff 2p(x, y) - p(x, x) - p(y, y) \leq \phi(x) - \phi(y) \\ &\iff 2(x \vee y) - x - y \leq -\frac{6}{x} + \frac{6}{y} \end{aligned}$$

According to this, $x \preceq y \Leftrightarrow (x \in \{1, 2, \dots, 6\} \text{ and } y = 1) \text{ or } (x = 3 \text{ and } y = 2) \text{ or } (x = y)$.

Our main result for single-valued mappings as follows.

Theorem 3.1. *Let (X, p) be a complete dualistic partial metric space, $\phi: X \rightarrow \mathbb{R}$ be a bounded below function and “ \preceq ” be the partial order induced by ϕ . If $f: X \rightarrow X$ is a $\tau((d_p)^s)$ -continuous nondecreasing function with $x_0 \preceq fx_0$ for some $x_0 \in X$, then f has a fixed point in X .*

Proof. Consider a point $x_0 \in X$ satisfying $x_0 \preceq fx_0$, then we define a sequence $\{x_n\}$ in X such that $x_n = fx_{n-1}$ for $n = 1, 2, \dots$. Then, since f is nondecreasing we have $x_0 \preceq x_1 \preceq x_2 \preceq \dots$, that is the sequence $\{x_n\}$ is nondecreasing. By the definition of “ \preceq ” we have $\dots \phi(x_2) \leq \phi(x_1) \leq \phi(x_0)$, that is, the sequence $\{\phi(x_n)\}$ is a nonincreasing sequence in \mathbb{R} . Since ϕ is bounded from below, $\{\phi(x_n)\}$ is convergent and hence it is Cauchy. That is, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\phi(x_n) - \phi(x_m)| < \varepsilon$. On the other hand, since $x_n \preceq x_m$, we have

$$2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \leq \phi(x_n) - \phi(x_m).$$

Therefore,

$$2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \leq \phi(x_n) - \phi(x_m) = |\phi(x_n) - \phi(x_m)| < \varepsilon. \quad (3.3)$$

On the other hand, since

$$d_p(x, y) = p(x, y) - p(x, x)$$

and

$$(d_p)^s(x, y) = \max\{d_p(x, y), d_p(y, x)\}$$

we have from (3.3),

$$(d_p)^s(x_n, x_m) < \varepsilon.$$

This shows that the sequence $\{x_n\}$ is Cauchy sequence in the metric space $(X, (d_p)^s)$, which is complete by Lemma 2.2. So there is $a \in X$ such that $\lim_{n \rightarrow \infty} (d_p)^s(x_n, a) = 0$.

Since f is $\tau((d_p)^s)$ -continuous, then we have $fa = a$. □

If we assume that $\phi(X)$ is compact in \mathbb{R} instead of boundedness of ϕ in Theorem 3.1, we can have the following theorem.

Theorem 3.2. *Let (X, p) be a complete dualistic partial metric space, $\phi: X \rightarrow \mathbb{R}$ be a function such that $\phi(X)$ is compact and “ \preceq ” be the partial order induced by ϕ . If $f: X \rightarrow X$ is a $\tau((d_p)^s)$ -continuous nondecreasing function with $x_0 \preceq fx_0$ for some $x_0 \in X$, then f has a fixed point in X .*

Now we give an example to illustrate our theorems. Also note that Theorem 2.1 is not applicable to this example.

Example 3.3. Let X , p and ϕ as in Example 3.2. Since $(X, (d_p)^s)$ is discrete metric space and every discrete metric spaces are complete, then (X, p) is complete. Suppose $f: X \rightarrow X$, $fx = x - 1$ for $x \in \{2, 3\}$ and $fx = x$ for $x \in \{1, 4, 5, \dots\}$. Then for $x_0 = 3$, we have $x_0 \preceq fx_0$. Also, since $x \preceq y \Leftrightarrow (x \in \{1, 2, \dots, 6\} \text{ and } y = 1) \text{ or } (x = 3 \text{ and } y = 2) \text{ or } (x = y)$, then it is obvious that $fx \preceq fy$, that is, f is nondecreasing with respect to \preceq . Again, since $(X, (d_p)^s)$ is discrete metric space, f is $\tau((d_p)^s)$ -continuous. Therefore all conditions of Theorem 3.1 or Theorem 3.2 are satisfied. Thus f has a fixed point. But since $p(f1, f4) = p(1, 4)$, then f is not contractive, therefore the results of [7] and [8] are not applicable to this example.

In the following we provide multi valued versions of the preceding theorem.

Let X be a topological space and \preceq be a partial order on X . Let 2^X denote the family of all nonempty subsets of X .

Definition 3.1 ([2]). Let A, B be two nonempty subsets of X , the relations between A and B are defined as follows:

- (r₁) If for every $a \in A$, there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.
- (r₂) If for every $b \in B$, there exists $a \in A$ such that $a \preceq b$, then $A \prec_2 B$.
- (r₃) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Remark 3.1 ([2]). \prec_1 and \prec_2 are different relations between A and B . For example, let $X = \mathbb{R}$, $A = \left[\frac{1}{2}, 1\right]$, $B = [0, 1]$, \preceq be usual order on X , then $A \prec_1 B$ but $A \not\prec_2 B$; if $A = [0, 1]$, $B = \left[0, \frac{1}{2}\right]$, then $A \prec_2 B$ while $A \not\prec_1 B$.

Remark 3.2 ([2]). \prec_1 , \prec_2 and \prec are reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0, 3]$, $B = [0, 1] \cup [2, 3]$, \preceq be usual order on X , then $A \prec B$ and $B \prec A$, but $A \neq B$. Hence, they are not partial orders on 2^X .

Definition 3.2 ([2]). A multi-valued operator $T: X \rightarrow 2^X$ is called order closed if for monotone sequences $\{u_n\}, \{v_n\} \subset X$, $u_n \rightarrow u_0, v_n \rightarrow v_0$ and $v_n \in Tu_n$ imply $v_0 \in Tu_0$.

We can define the order closed operator on ordered dualistic partial metric spaces by the following way.

Definition 3.3. Let (X, p) be a ordered dualistic partial metric space. A multi-valued operator $T: X \rightarrow 2^X$ is called p -order closed if for monotone sequences $\{u_n\}, \{v_n\} \subset X$, $\lim_{n \rightarrow \infty} p(u_n, u_0) = p(u_0, u_0)$, $\lim_{n \rightarrow \infty} p(v_n, v_0) = p(v_0, v_0)$ and $v_n \in Tu_n$ imply $v_0 \in Tu_0$.

Now we prove some fixed point theorems for multi-valued maps.

Theorem 3.3. Let (X, p) be a complete dualistic partial metric space, $\phi: X \rightarrow \mathbb{R}$ be a bounded below function and " \preceq " be the partial order induced by ϕ . $F: X \rightarrow 2^X$ is p -order closed operator with $\{x_0\} \prec_1 Fx_0$ for some $x_0 \in X$. If

$$(\forall) x, y \in X, \quad x \preceq y \implies Fx \prec_1 Fy$$

(that is, F is nondecreasing with respect to \prec_1), then F has a fixed point in X .

Proof. Since Fx is nonempty for all $x \in X$, then there exists $x_1 \in Fx_0$ such that $x_0 \preceq x_1$. Now since $Fx_0 \prec_1 Fx_1$, then there exists $x_2 \in Fx_1$ such that $x_1 \preceq x_2$. Continue this process, we will get a nondecreasing sequence $\{x_n\}$, which satisfies $x_{n+1} \in Fx_n$. By the definition of “ \preceq ”, we have $\dots \leq \phi(x_2) \leq \phi(x_1) \leq \phi(x_0)$, that is, the sequence $\{\phi(x_n)\}$ is a nonincreasing sequence in \mathbb{R} . Since ϕ is bounded from below, $\{\phi(x_n)\}$ is convergent and hence it is Cauchy. That is, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\phi(x_m) - \phi(x_n)| < \varepsilon$. On the other hand, since $x_n \preceq x_m$, we have

$$2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \leq \phi(x_n) - \phi(x_m).$$

Therefore,

$$2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \leq \phi(x_n) - \phi(x_m) = |\phi(x_n) - \phi(x_m)| < \varepsilon.$$

On the other hand, since

$$d_p(x, y) = p(x, y) - p(x, x)$$

and

$$(d_p)^s(x, y) = \max\{d_p(x, y), d_p(y, x)\}$$

we have from,

$$(d_p)^s(x_n, x_m) < \varepsilon.$$

This shows that the sequence $\{x_n\}$ is Cauchy sequence in the metric space $(X, (d_p)^s)$, which is complete by Lemma 2.2. So there is $a \in X$ such that $\lim_{n \rightarrow \infty} (d_p)^s(x_n, a) = 0$.

Note that, from Lemma 2.2, we obtain

$$p(a, a) = \lim_{n \rightarrow \infty} p(a, x_n) = \lim_{m, n \rightarrow \infty} p(x_n, x_m).$$

Consequently, we have $a \in Fa$ since F is p -order closed and $x_{n+1} \in Fx_n$. □

Similarly, we can prove the following theorem.

Theorem 3.4. *Let (X, p) be a complete dualistic partial metric space, $\phi: X \rightarrow \mathbb{R}$ be a bounded above function and “ \preceq ” be the partial order induced by ϕ . $F: X \rightarrow 2^X$ is p -order closed operator with $Fx_0 \prec_2 \{x_0\}$ for some $x_0 \in X$. If*

$$(\forall) x, y \in X, \quad x \preceq y \implies Fx \prec_2 Fy$$

(that is, F is nondecreasing with respect to \prec_2), then F has a fixed point in X .

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