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NULL LAGRANGIAN FORMS AND EULER-LAGRANGE PDEs

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Abstract. The paper offers fresh light and novelties on the variational problems whose Euler-Lagrange PDEs are identically satisfied. §1 underlines that the Jacobian is a null Lagrangian. §2 proves an equivalence theorem between multiple and curvilinear integral. §4 studies the multi-time Lagrange 1-forms and their primitives as path independent curvilinear integrals. §5 finds the Euler-Lagrange PDEs associated to path independent curvilinear functionals. §6 gives the hyperbolic primitive of a Lagrangian. §7 describes the connection between single-time Lagrangians and the single-time total derivatives. §8 shows that a multi-time null Lagrangian is a multi-time total divergence. §9 proves that a multi-time null Lagrangian is a multi-time total derivative.

1. JACOBIAN AS NULL LAGRANGIAN

The term *null Lagrangian* [1], [3], [4] pertains to a Lagrangian L whose integral over any domain $\Omega \subset \mathbb{R}^m$ can be reduced via integration by parts to an integral over the boundary $\partial\Omega$. An important special case is furnished by the Jacobian determinant $L = J(x)(t)$ associated to a diffeomorphism $x\colon \mathbb{R}^m \to \mathbb{R}^m$, which is used in the volume functional

$$V(x(\cdot)) = \int_\Omega J(x)(t)dt^1 \dots dt^m.$$

Null Lagrangians owe much of their importance to recent advances in the calculus of variations (polyconvex energy integrals), nonlinear PDEs (compensated compactness), geometric function theory (quasiconformal deformations) and some fields of applied mathematics: nonlinear elasticity, material science and crystals (rank-one connections). Adding null Lagrangian to any integrand does not affect the variational Euler-Lagrange equation. The present paper offers fresh light on these topics involving curvilinear functionals (see also [2]).

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2. EQUIVALENCE BETWEEN MULTIPLE AND CURVILINEAR INTEGRAL

Many problems reduce to extremizing a *multi-time cost functional.* On the other hand, the multi-time cost functionals can be introduced in at least two ways:

- either using a path independent curvilinear integral,

$$P(x(\cdot)) = \int_{\Gamma_{0,t_0}} X^0_\beta(t, x(t), x_\gamma(t))dt^\beta, \quad \beta = 1, \dots, m,$$

where Γ_{0,t_0} is an arbitrary C^1 curve joining the points 0 and t_0, the *running cost* $\omega = X^0_\beta(t, x(t), x_\gamma(t))dt^\beta$ is a closed (completely integrable) 1-form (Lagrangian 1-form);

- or using a multiple integral,

$$Q(x(\cdot)) = \int_{\Omega_{0,t_0}} X(t, x(t), x_\gamma(t))dt^1 \dots dt^m,$$

where the *running cost* $X(t, x(t), x_\gamma(t))$ is a continuous function (Lagrangian).

Let us ask if the functional P is equivalent to the functional Q.

Theorem 2.1. *The multiple integral*

$$I(t_0) = \int_{\Omega_{0,t_0}} X(t)dt^1 \dots dt^m,$$

were X is a continuous function, is equivalent to the curvilinear integral

$$J(t_0) = \int_{\Gamma_{0,t_0}} X^0_\beta(t)dt^\beta,$$

where $\omega = X^0_\beta(t, x(t), x_\gamma(t))dt^\beta$ is a closed (completely integrable) 1-form and the functions X^0_β have partial derivatives of order $m-1$ and

$$\frac{\partial^{m-1} X^0_\beta}{\partial t^1 \dots \hat{\partial t^\alpha} \dots \partial t^m} = X,$$

where the symbol "ˆ" posed over ∂t^α designates that ∂t^α is omitted.

Proof. The multiple integral $I(t_0)$ suggests to introduce a new coordinate

$$x^0(t) = \int_{\Omega_{0,t}} X(s)ds^1 \dots ds^m, \quad t \in \Omega_{0,t_0},\ x^0(t_0) = I(t_0).$$

Taking

$$X^0_\beta(t, x(t), x_\gamma(t)) = \frac{\partial x^0}{\partial t^\beta}(t),$$

we can write $x^0(t)$ as the curvilinear integral

$$x^0(t) = \int_{\Gamma_{0,t}} X^0_\alpha(s)ds^\alpha,\ x^0(t_0) = J(t_0),$$

where $\Gamma_{0,t}$ is an arbitrary C^1 curve joining the points 0 and t in Ω_{0,t_0}. Also

$$\frac{\partial^{m-1} X^0_\beta}{\partial t^1 \dots \hat{\partial t^\beta} \dots \partial t^m} = \frac{\partial^m x^0}{\partial t^1 \dots \partial t^m} = X.$$

Conversely, the curvilinear integral $J(t_0)$ suggests to define a new coordinate by

$$x^0(t) = \int_{\Gamma_{0,t}} X^0_\beta(s) ds^\beta,\ x^0(t_0) = J(t_0),$$

where $\omega = X^0_\beta(s)ds^\beta$ is a closed (completely integrable) 1-form. Since $X^0_\beta = \dfrac{\partial x^0}{\partial t^\beta}$, we can define

$$X = \frac{\partial^{m-1} X^0_\beta}{\partial t^1 \dots \hat{\partial t^\beta} \dots \partial t^m} = \frac{\partial^m x^0}{\partial t^1 \dots \partial t^m}.$$

Then the new function can be written as

$$x^0(t) = \int_{\Omega_{0,t}} X(s) ds^1 \dots ds^m, \quad t \in \Omega_{0,t_0},\ x^0(t_0) = I(t_0).$$

□

Using the above theorem, one can convert variational problems with multiple integrals into problems with curvilinear integrals and reciprocally. On this way there naturally arise the following questions (see also, [5]-[10]):

(1) In what conditions *the curvilinear indefinite integral* (*primitive*) of a closed Lagrangian 1-form is also a Lagrangian?

(2) In what conditions *the hyperbolic indefinite integral* (*primitive*) of a Lagrangian is a Lagrangian too?

3. LAGRANGE 1-FORMS OF THE FIRST ORDER AND THEIR PRIMITIVES AS SIMPLE INTEGRALS

A Lagrange 1-form of the first order on the jets space $J^1(\mathbb{R}, M)$ has the form

$$\omega = L(t, x(t), \dot{x}(t))dt + M_i(t, x(t), \dot{x}(t))dx^i + N_i(t, x(t), \dot{x}(t))d\dot{x}^i,$$

where L, M_i, N_i are Lagrangians of the first order. The pullback

$$x^*\omega = (L + M_i \dot{x}^i + N_i \ddot{x}^i)dt$$

is a Lagrange 1-form of the second order on M. The coefficient

$$L + M_i \dot{x}^i + N_i \ddot{x}^i$$

is a second order Lagrangian, which is linear in accelerations. To the form ω one attachs the Pfaff equation $\omega = 0$ and the differential equation

$$L + M_i \dot{x}^i + N_i \ddot{x}^i = 0.$$

Generally, the primitive of a Lagrange 1-form $L(t,x(t),\dot x(t))dt$ can be written either as (definite integral)

$$\phi(t)=\int_{t_0}^{t}L(s,x(s),\dot x(s))ds,\ \ \phi(t_0)=0$$

or as (differential equation)

$$\dot\phi(t)=L(t,x(t),\dot x(t)),\ \ \phi(t_0)=0.$$

If would exist a Lagrangian-like primitive

$$K(t,x(t),\dot x(t))=\int_{t_0}^{t}L(s,x(s),\dot x(s))ds,\ \ K(x(t_0),\dot x(t_0),t_0)=0$$

or in another form $\frac{d}{dt}K=L$, then it arises the relation

$$\frac{\partial K}{\partial x^i}(t,x(t),\dot x(t))\dot x^i(t)+\frac{\partial K}{\partial \dot x^i}(t,x(t),\dot x(t))\ddot x^i(t)+\frac{\partial K}{\partial t}(t,x(t),\dot x(t))=L(t,x(t),\dot x(t)).$$

This relation is fulfilled by any function $x(t)$ iff $\frac{\partial K}{\partial \dot x^i}=0$. It results $K=K(t,x(t))$, and L is a linear function in $\dot x^i$.

4. LAGRANGE 1-FORMS OF THE FIRST ORDER AND THEIR PRIMITIVES AS CURVILINEAR INTEGRALS

A Lagrange 1-form of the first order on the jets space $J^1(T,M)$ has the form

$$\omega=L_\alpha(t,x(t),x_\gamma(t))dt^\alpha+M_i(t,x(t),x_\gamma(t))dx^i+N_i^\beta(t,x(t),x_\gamma(t))dx^i_\beta,$$

where L_α, M_i, N_i^β are Lagrangians of the first order and $x_\gamma(t)=\frac{\partial x}{\partial t^\gamma}(t)$. The pull-back

$$x^*\omega=(L_\alpha+M_ix^i_\alpha+N_i^\beta x^i_{\beta\alpha})dt^\alpha$$

is a Lagrange 1-form of the second order on M. The coefficients

$$L_\alpha+M_ix^i_\alpha+N_i^\beta x^i_{\beta\alpha}$$

are second order Lagrangians, which are linear in the partial accelerations. To the form ω one attachs the Pfaff equation $\omega=0$ and the partial differential equations

$$L_\alpha+M_ix^i_\alpha+N_i^\beta x^i_{\beta\alpha}=0.$$

Let $L_\beta(t,x(t),x_\gamma(t))dt^\beta$ be a closed Lagrange 1-form (completely integrable), i.e. $D_\beta L_\alpha=D_\alpha L_\beta$, or explicitely

$$\frac{\partial L_\beta}{\partial x^i}\frac{\partial x^i}{\partial t^\alpha}+\frac{\partial L_\beta}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\alpha}+\frac{\partial L_\beta}{\partial t^\alpha}=\frac{\partial L_\alpha}{\partial x^i}\frac{\partial x^i}{\partial t^\beta}+\frac{\partial L_\alpha}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\beta}+\frac{\partial L_\alpha}{\partial t^\beta}.$$

We remind that a closed 1-form in a simple-connected domain is an exact one. Generally, its primitive has the form

$$\phi(t)=\int_{\Gamma_{t_0,t}} L_\alpha\left(s,x(s),x_\gamma(s)\right)ds^\alpha,\ \ \phi(t_0)=0,$$

i.e.

$$\frac{\partial\phi}{\partial t^\alpha}(t)=L_\alpha\left(t,x(t),x_\gamma(t)\right),\ \ \phi(t_0)=0.$$

If would exist a Lagrangian-like primitive

$$L(t,x(t),x_\gamma(t))=\int_{\Gamma_{t_0,t}} L_\alpha\left(s,x(s),x_\gamma(s)\right)ds^\alpha,\ \ L(t_0,x(t_0),x_\gamma(t_0))=0$$

or in another form $D_\alpha L=L_\alpha$ (the above pullback is the given closed 1-form), then it arises the relation

$$\frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\beta}+\frac{\partial L}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\beta}+\frac{\partial L}{\partial t^\beta}=L_\beta,$$

which can be understood as a completely integrable system of PDE (of the second order) with the unknown function $x(t)$, too.

A smooth *Lagrangian* $L(t,x(t),x_\gamma(t))$, $t\in\mathbb{R}^m_+$ produces two smooth closed (completely integrable) 1-forms:

- the differential

$$dL=\frac{\partial L}{\partial x^i}dx^i+\frac{\partial L}{\partial x^i_\gamma}dx^i_\gamma+\frac{\partial L}{\partial t^\gamma}dt^\gamma$$

of components $\left(\frac{\partial L}{\partial t^\gamma},\frac{\partial L}{\partial x^i},\frac{\partial L}{\partial x^i_\gamma}\right)$, with respect to the basis $(dt^\gamma,dx^i,dx^i_\gamma)$;

- the restriction of dL to $(t,x(t),x_\gamma(t))$, i.e. the pullback

$$dL|_{(t,x(t),x_\gamma(t))}=\left(\frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\beta}+\frac{\partial L}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\beta}+\frac{\partial L}{\partial t^\beta}\right)dt^\beta,$$

of components (which contain partial accelerations)

$$D_\beta L=\frac{\partial L}{\partial x^i}(t,x(t),x_\gamma(t))\frac{\partial x^i}{\partial t^\beta}(t)+\frac{\partial L}{\partial x^i_\gamma}(t,x(t),x_\gamma(t))\frac{\partial x^i_\gamma}{\partial t^\beta}(t)+\frac{\partial L}{\partial t^\beta}(t,x(t),x_\gamma(t)),$$

with respect to the basis dt^β (for other ideas, see [5]-[10]).

5. THE EXTREMALS OF THE FUNCTIONALS REPRESENTED BY PATH-INDEPENDENT CURVILINEAR INTEGRALS

Let Γ_{t_0,t_1} be an arbitrary piecewise C^1-curve joining the diagonal points t_0 and t_1 of the parallelepiped $\Omega_{t_0,t_1}\subset\mathbb{R}^m_+$. Let us introduce a new problem in the calculus of variations asking for a m-sheet $x^*(\cdot) : \Omega_{t_0,t_1} \to \mathbb{R}^n$ to minimize *the functional represented by path-independent curvilinear integral (action)*

$$J(x(\cdot)) = \int_{\Gamma_{0,t_0}} L_\beta(t, x(t), x_\gamma(t))dt^\beta,\ \beta = 1, \dots, m,$$

such that the functions $x(\cdot)$ have to satisfy the boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$ or $x(t)|_{\partial\Omega_{t_0,t_1}} =$ given, and variations (functions) constrained by boundary conditions and by closedness conditions (completely integrability) for the Lagrange 1-form.

Fundamental problem. Characterize the function $x^*(\cdot)$ which solves the variational problem associated to functional J.

Theorem 5.1. *Suppose that it exists a Lagrangian $L(t, x(t), x_\gamma(t))$ with the property $D_\beta L = L_\beta$.*

1) *If the m-sheet $x^*(\cdot)$ is an extremal for L, then it is an extremal for the differential dL also.*

2) *If the m-sheet $x^*(\cdot)$ minimizes the functional $J(x(\cdot))$, then $x^*(\cdot)$ is a solution of the multi-time PDE*

$$\frac{\partial L}{\partial x^i} - D_\gamma \frac{\partial L}{\partial x^i_\gamma} = a_i,\ i = 1, \dots, n,\ \gamma = 1, \dots, m, \qquad (E-L)_2$$

which satisfies boundary conditions, where a_i are arbitrary constants.

The second part of this theorem shows that if the system $(E - L)_2$ of PDSs has solutions, then the minimizing function for the functional J (supposing it exists) lies between these solutions.

Proof. 1) First, we have

$$0 = d\left(\frac{\partial L}{\partial x^i} - D_\gamma \frac{\partial L}{\partial x^i_\gamma}\right) = \frac{\partial(dL)}{\partial x^i} - D_\gamma \frac{\partial(dL)}{\partial x^i_\gamma}.$$

For the components, this means

$$\frac{\partial}{\partial x^i}\left(\frac{\partial L}{\partial x^j}\right) - D_\gamma \frac{\partial}{\partial x^i_\gamma}\left(\frac{\partial L}{\partial x^j}\right) = 0$$

$$\frac{\partial}{\partial x^i}\left(\frac{\partial L}{\partial x^j_\alpha}\right) - D_\gamma \frac{\partial}{\partial x^i_\gamma}\left(\frac{\partial L}{\partial x^j_\alpha}\right) = 0$$

$$\frac{\partial}{\partial x^i}\left(\frac{\partial L}{\partial t^\alpha}\right) - D_\gamma \frac{\partial}{\partial x^i_\gamma}\left(\frac{\partial L}{\partial t^\alpha}\right) = 0.$$

2) The following equalities lied us to the result.

$$0 = \frac{\partial(dL)}{\partial x^i} - D_\gamma \frac{\partial(dL)}{\partial x^i_\gamma} = d\left(\frac{\partial L}{\partial x^i} - D_\gamma \frac{\partial L}{\partial x^i_\gamma}\right).$$

□

Theorem 5.2. *If the m-sheet $x^*(\cdot)$ minimizes the functional $J(x(\cdot))$, then $x^*(\cdot)$ is a solution of the multi-time PDE*

$$\frac{\partial L_\beta}{\partial x^i} - D_\gamma \frac{\partial L_\beta}{\partial x^i_\gamma} = 0,\ \beta,\, \gamma = 1, \dots, m, \qquad (E-L)_3$$

which satisfies boundary conditions.

The theorem shows that if the system $(E-L)_3$ of PDSs has solutions, then the minimizing function for the functional J (supposing it exists) lies between these solutions.

Proof. Let us consider that $x(t)$ is a solution of the above problem. It has to satisfy the complete integrability conditions of the 1-form $L_\beta(t, x(t), x_\gamma(t))dt^\beta$, i.e.

$$\frac{\partial L_\beta}{\partial x^i}\frac{\partial x^i}{\partial t^\alpha} + \frac{\partial L_\beta}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\alpha} + \frac{\partial L_\beta}{\partial t^\alpha} = \frac{\partial L_\alpha}{\partial x^i}\frac{\partial x^i}{\partial t^\beta} + \frac{\partial L_\alpha}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\beta} + \frac{\partial L_\alpha}{\partial t^\beta}.$$

Construct, nearly $x(t)$ another function having the form $x(t) + \varepsilon h(t)$, with $h(t_0) = 0, h(t_1) = 0$. Here ε is a "little" parameter, and h is a "little" variation. The functional becomes a function of ε, i.e. an integral dependent on parameter,

$$J(\varepsilon) = \int_{\Gamma_{t_0,t_1}} L_\beta\left(t, x(t) + \varepsilon h(t), x_\gamma(t) + \varepsilon h_\gamma(t)\right) dt^\beta.$$

We accept the variation h satisfies the complete integrability conditions on 1-form

$$L_\beta\left(t, x(t) + \varepsilon h(t), x_\gamma(t) + \varepsilon h_\gamma(t)\right) dt^\beta.$$

These conditions add PDEs

$$\frac{\partial L_\beta}{\partial x^i}\frac{\partial h^i}{\partial t^\alpha} + \frac{\partial L_\beta}{\partial x^i_\gamma}\frac{\partial h^i_\gamma}{\partial t^\alpha} = \frac{\partial L_\alpha}{\partial x^i}\frac{\partial h^i}{\partial t^\beta} + \frac{\partial L_\alpha}{\partial x^i_\gamma}\frac{\partial h^i_\gamma}{\partial t^\beta},$$

which shows that the set of the functions $h(t)$ is a vector space and the set of the functions $x(t) + \varepsilon h(t)$ is an affine space. One imposes

$$0 = \frac{d}{d\varepsilon} J(\varepsilon)\Big|_{\varepsilon=0} = (\cdots) = \int_{\Gamma_{t_0,t_1}} \left(\frac{\partial L_\beta}{\partial x^j} h^j + \frac{\partial L_\beta}{\partial x^j_\gamma} h^j_\gamma\right) dt^\beta$$

$$= \mathrm{BT} + \int_{\Gamma_{t_0,t_1}} \left(\frac{\partial L_\beta}{\partial x^j} - D_\gamma \frac{\partial L_\beta}{\partial x^j_\gamma}\right) h^j\, dt^\beta,$$

where D_γ is the total derivative operator acting by the rule

$$D_\gamma\left(\frac{\partial L_\beta}{\partial x^j_\gamma} h^j\right) = h^j D_\gamma\left(\frac{\partial L_\beta}{\partial x^j_\gamma}\right) + \frac{\partial L_\beta}{\partial x^j_\gamma} D_\gamma h^j.$$

Let us add the hypothesis

$$D_\gamma\left(h^i \frac{\partial}{\partial x^i_\gamma} L_\beta\right) = D_\beta\left(h^i \frac{\partial}{\partial x^i_\gamma} L_\gamma\right),$$

which restricts the vector space of the variations $h(t)$ to a subspace. Then on boundary we can evaluate

$$BT = \int_{\Gamma_{t_0,t_1}} D_\gamma \left(\frac{\partial L_\beta}{\partial x^j_\gamma} h^j \right) dt^\beta = \int_{\Gamma_{t_0,t_1}} D_\beta \left(\frac{\partial L_\gamma}{\partial x^j_\gamma} h^j \right) dt^\beta = \frac{\partial L_\gamma}{\partial x^j_\gamma} h^j \Big|_{t_0}^{t_1}.$$

The terms BT vanish since $h(t_0) = 0, h(t_1) = 0$. It remains

$$0 = \int_{\Gamma_{t_0,t_1}} \left(\frac{\partial L_\beta}{\partial x^j} - D_\gamma \frac{\partial L_\beta}{\partial x^j_\gamma} \right) h^j \, dt^\beta.$$

Since the curve Γ_{t_0,t_1} is arbitrary, we find the PDEs system involved in theorem. □

For other ideas regarding the path independent curvilinear functional, see [5]-[10].

6. THE HYPERBOLIC PRIMITIVE OF A LAGRANGIAN

Let be the parallelepiped $\Omega_{t_0,t} \subset \mathbb{R}^m_+$ and the smooth *Lagrangian* $L(t, x(t), x_\gamma(t))$, $t \in \mathbb{R}^m_+$. The hyperbolic primitive of L has the form

$$\Phi(t) = \int_{\Omega_{t_0,t}} L\left(s, x(s), x_\gamma(s)\right) ds^1 \dots ds^m, \ \Phi(t_0) = 0,$$

i.e.

$$\frac{\partial^m \Phi}{\partial t^1 \dots \partial t^m}(t) = L\left(t, x(t), x_\gamma(t)\right), \ \Phi(t_0) = 0.$$

If would exist a Lagrangian-like hyperbolic primitive

$$K(t, x(t), x_\gamma(t)) = \int_{\Omega_{t_0,t}} L\left(s, x(s), x_\gamma(s)\right) ds^1 \dots ds^m, \ K(t_0, x(t_0), x_\gamma(t_0)) = 0,$$

then $D_1 \dots D_m K = L$.

7. NULL LAGRANGIANS AND THE TOTAL DERIVATIVE

There exist Lagrangians whose Euler-Lagrange ODE are satisfied identically. If this case, each functions may be an extremal one and such a Lagrangian is called *null Lagrangian.* The definition of null Lagrangians does not depend upon the choosing of coordinate system. Generally, a Lagrangian having the form $L(t, x(t), \dot{x}(t)) = D_t K(t, x(t))$ is a null one because

$$\int_{t_0}^{t_1} L(t, x(t), \dot{x}(t)) dt = K(t, x(t))\big|_{t_0}^{t_1},$$

i.e. the integral depends upon the values of the function $x(t)$ taken on the boundary only and it is not affected by variations $h(t)$. Fortunately, these only are null Lagrangians.

Theorem 7.1 ([3]). *A Lagrangian $L(t, x(t), \dot{x}(t))$, $t \in \mathbb{R}_+$ is a null one if it is a total derivative.*

Proof. Suppose that it exists a Lagrangian $K = K(t, x(t), \dot{x}(t))$ with $L = D_t K$. Obviously, L is a second order Lagrangian. The Euler-Lagrange equations

$$\frac{\partial L}{\partial x^j} - D_t \frac{\partial L}{\partial \dot{x}^j} + D_t D_t \frac{\partial L}{\partial \ddot{x}^j} = 0$$

are satisfied identically, i.e. $L = D_t K$ is a null Lagrangian.

Conversely, suppose the Euler-Lagrange equations of the Lagrangian $L(t, x(t), \dot{x}(t))$, $t \in \mathbb{R}_+$ are satisfied identically.

Construct the function $f(\varepsilon) = L(t, \varepsilon x(t), \varepsilon \dot{x}(t))$ and note $u(t) = \varepsilon x(t)$. Calculate the derivative

$$\frac{d}{d\varepsilon} f = x^i \frac{\partial L}{\partial u^i} + \dot{x}^i \frac{\partial L}{\partial \dot{u}^i}$$

or, using the total derivative of a product,

$$\frac{d}{d\varepsilon} f = x^i \left(\frac{\partial L}{\partial u^i} - D_t \frac{\partial L}{\partial \dot{u}^i} \right) + D_t \left(x^i \frac{\partial L}{\partial \dot{u}^i} \right) = D_t \left(x^i \frac{\partial L}{\partial \dot{u}^i} \right).$$

Integrating we have

$$L(t, x(t), x_\gamma(t) - L(t, 0, 0) = D_t \int_0^1 x^i \frac{\partial L}{\partial \dot{u}^i} d\varepsilon.$$

Moreover, it exists a function k with $D_t k = L(t, 0, 0)$. So

$$L(t, x(t), \dot{x}(t)) = D_t k + D_t K,$$

where $K = \displaystyle\int_0^1 x^i \frac{\partial L}{\partial \dot{u}^i} d\varepsilon$. □

8. NULL LAGRANGIANS AND THE TOTAL DIVERGENCE

Theorem 8.1. *The tensor*

$$T^\gamma_{\beta i} = \frac{\partial L}{\partial x^i} \delta^\gamma_\beta - D_\beta \frac{\partial L}{\partial x^i_\gamma}$$

represents a conservation law with respect to Euler-Lagrange evolution

$$\frac{\partial L}{\partial x^i} - D_\gamma \frac{\partial L}{\partial x^i_\gamma} = 0.$$

Proof. Let us prove that Euler-Lagrange derivative is the curvilinear primitive of a divergence. Indeed, the tensor $T^\gamma_{\beta i}$ satisfies

$$D_\gamma T^\gamma_{\beta i} = D_\beta \left(\frac{\partial L}{\partial x^i} - D_\gamma \frac{\partial L}{\partial x^i_\gamma} \right).$$

□

There exist Lagrangians whose Euler-Lagrange equations are satisfied identically. If this case, each functions may be an extremal one and such a Lagrangian is called *null Lagrangian.* The definition of null Lagrangians does not depend upon the choosing of coordinate system.

Let be $S = \partial\Omega$ and n the normal versor to S. If a Lagrangian can be written as a total divergence $L(t, x(t), \dot{x}(t)) = Div\, P(t, x(t))$, then by the theorem of the divergence we obtain

$$\int_\Omega L\, dt^1 \dots dt^m = \int_\Omega Div\, P\, dt^1 \dots dt^m = \int_{\partial\Omega} P \cdot n\, dS.$$

In other words, the Euler-Lagrange equations are satisfied identically because the functional depends upon the values of the function $x(t)$ taken on the boundary only and it is not affected by variations $h(t)$. So the Lagrangian $L = Div\, P$ is a null one. Fortunately, only the Lagrangians as a total divergence are null Lagrangians.

Theorem 8.2 ([3]). *A Lagrangian* $L(t, x(t), x_\gamma(t))$, $t \in \mathbb{R}^m_+$ *is a null one if it is a total divergence.*

Proof. Suppose it exists a vectorial field $P = (P^\gamma)$, $P^\gamma = P^\gamma(t, x(t), x_\gamma(t))$ with $L = Div\, P = D_\gamma P^\gamma$. Obviously L is a second order Lagrangian. The Euler-Lagrange equations

$$\frac{\partial L}{\partial x^j} - D_\lambda \frac{\partial L}{\partial x^j_\lambda} + D_\lambda D_\gamma \frac{\partial L}{\partial x^j_{\lambda\gamma}} = 0$$

are satisfied identically. So the Lagrangian $L = Div\, P$ is a null one.

Conversely, suppose the Euler-Lagrange equations of the Lagrangian $L(t, x(t), \dot{x}(t))$, $t \in \mathbb{R}_+$ are satisfied identically.

Construct the function $f(\varepsilon) = L(t, \varepsilon x(t), \varepsilon \dot{x}(t))$ and note $u(t) = \varepsilon x(t)$. Calculate the derivative

$$\frac{d}{d\varepsilon} f = x^i \frac{\partial L}{\partial u^i} + x^i_\gamma \frac{\partial L}{\partial u^i_\gamma}$$

and using the divergence of a product

$$\frac{d}{d\varepsilon} f = x^i \left(\frac{\partial L}{\partial u^i} - D_\gamma \frac{\partial L}{\partial u^i_\gamma} \right) + D_\gamma \left(x^i \frac{\partial L}{\partial u^i_\gamma} \right) = D_\gamma \left(x^i \frac{\partial L}{\partial u^i_\gamma} \right).$$

Integrating, we found

$$L(t, x(t), x_\gamma(t) - L(t, 0, 0) = D_\gamma \int_0^1 x^i \frac{\partial L}{\partial u^i_\gamma} d\varepsilon.$$

Moreover, it exists a vector p with $\operatorname{div} p = L(t, 0, 0)$. So

$$L(t, x(t), x_\gamma(t)) = div\, p + Div\, P,$$

where $P = (P^\gamma)$, $P^\gamma = \displaystyle\int_0^1 x^i \frac{\partial L}{\partial u^i_\gamma} d\varepsilon.$ □

Proposition 8.1. *A null Lagrangian L is the primitive of the closed Lagrange 1-form*

$$\left(\frac{\partial L}{\partial t^\alpha} + D_\gamma\left(\frac{\partial x^i}{\partial t^\alpha}\frac{\partial L}{\partial x^i_\gamma}\right)\right) dt^\alpha.$$

Proof. Calculating we have successively

$$\begin{aligned} dL(t, x(t), x_\gamma(t)) &= \left(\frac{\partial L}{\partial t^\alpha} + \frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\alpha} + \frac{\partial L}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\alpha}\right) dt^\alpha \\ &= \frac{\partial L}{\partial t^\alpha}dt^\alpha + \left(\frac{\partial L}{\partial x^i} - D_\gamma\frac{\partial L}{\partial x^i_\gamma}\right)\frac{\partial x^i}{\partial t^\alpha}dt^\alpha + D_\gamma\left(\frac{\partial x^i}{\partial t^\alpha}\frac{\partial L}{\partial x^i_\gamma}\right)dt^\alpha \\ &= \left(\frac{\partial L}{\partial t^\alpha} + D_\gamma\left(\frac{\partial x^i}{\partial t^\alpha}\frac{\partial L}{\partial x^i_\gamma}\right)\right) dt^\alpha. \end{aligned}$$

□

Open problem. What we can say about the Legendre duality when the Lagrangian is null?

9. NULL LAGRANGE FORMS AND TOTAL DERIVATIVES

In the multi-time case, there exist closed Lagrange 1-forms whose Euler-Lagrange equations are satisfied identically. If this case, each functions may be an extremal one and such a Lagrange 1-form is called *null*. The definition of null Lagrange 1-forms does not depend upon the choosing of coordinate system.

If a Lagrange 1-form can be written as $L_\alpha = D_\alpha L$, then

$$\begin{aligned} \int_{\Gamma_{t_0t_1}} L_\alpha dt^\alpha &= \int_{\Gamma_{t_0t_1}} D_\alpha L\, dt^\alpha \\ &= \int_{\tilde{\Gamma}_{t_0,x(t_0),x_\gamma(t_0);t_1,x(t_1),x_\gamma(t_1)}} \frac{\partial L}{\partial t^\alpha}dt^\alpha + \frac{\partial L}{\partial x^i}dx^i + \frac{\partial L}{\partial x^i_\gamma}dx^i_\gamma \\ &= L(t, x(t), x_\gamma(t))\Big|_{t_0}^{t_1}. \end{aligned}$$

In this case the functional depends upon the values of the function $x(t)$ taken on the boundary only and it is not affected by variations $h(t)$. Hence the 1-form $L_\alpha = D_\alpha L$ is a null one. Fortunately, these only are null Lagrange 1-forms.

Theorem 9.1. *A closed Lagrange 1-form $L_\alpha(t, x(t), x_\gamma(t))dt^\alpha$, $t \in \mathbb{R}^m_+$ is a null one if it is a total divergence.*

Proof. Suppose it exists a Lagrangian $L(t, x(t), x_\gamma(t))$ with $L_\alpha = D_\alpha L$. Then $L_\alpha dt^\alpha$ is a Lagrange 1-form of the second order. The Euler-Lagrange equations

$$\frac{\partial L_\alpha}{\partial x^j} - D_\lambda\frac{\partial L_\alpha}{\partial x^j_\lambda} + D_\lambda D_\gamma\frac{\partial L_\alpha}{\partial x^j_{\lambda\gamma}} = 0$$

are satisfied identically.

Conversely, suppose the Euler-Lagrange equations of the closed Lagrange 1-form $L_\alpha(t, x(t), x_\gamma(t))$, $t \in \mathbb{R}^m_+$ are satisfied identically. Construct the functions $f_\alpha(\varepsilon) = L_\alpha(t, \varepsilon x(t), \varepsilon x_\gamma(t))$ and note $u(t) = \varepsilon x(t)$. Calculate the derivatives

$$\frac{d}{dt} f_\alpha = x^i \frac{\partial L_\alpha}{\partial u^i} + x^i_\gamma \frac{\partial L_\alpha}{\partial u^i_\gamma}$$

and using the divergence of a product

$$\frac{d}{dt} f_\alpha = x^i \left(\frac{\partial L_\alpha}{\partial u^i} - D_\gamma \frac{\partial L_\alpha}{\partial u^i_\gamma} \right) + D_\gamma \left(x^i \frac{\partial L_\alpha}{\partial u^i_\gamma} \right) = D_\gamma \left(x^i \frac{\partial L_\alpha}{\partial u^i_\gamma} \right).$$

Integrating, we found

$$L_\alpha(t, x(t), x_\gamma(t) - L_\alpha(t, 0, 0) = D_\gamma \int_0^1 x^i \frac{\partial L_\alpha}{\partial u^i_\gamma} d\varepsilon = D_\gamma P^\gamma_\alpha.$$

Moreover, it exists a function p with $D_\alpha p = L_\alpha(t, 0, 0)$. So

$$L_\alpha(t, x(t), x_\gamma(t)) = D_\alpha p + D_\gamma P^\gamma_\alpha.$$

□

Corollary 9.1. *The total divergence of a tensor of type* $(1, 1)$ *is the total derivative of a function.*

Proof. Each Lagrange 1-form of as $D_\alpha K \, dt^\alpha$ (total derivative) is a null one. By virtue of the previous theorem, this total derivative is in the same time a total divergence. □

Proposition 9.1. *If* $L_\alpha dt^\alpha$ *is a null Lagrange* 1*-form, then the Lagrange* 1*-forms*

$$\left(\frac{\partial L_\alpha}{\partial t^\beta} + D_\gamma \left(\frac{\partial x^i}{\partial t^\beta} \frac{\partial L_\alpha}{\partial x^i_\gamma} \right) \right) dt^\alpha$$

are exact.

Proof. The following equalities lied us to the result.

$$\begin{aligned}
dL_\beta(t, x(t), x_\gamma(t)) &= \left(\frac{\partial L_\beta}{\partial t^\alpha} + \frac{\partial L_\beta}{\partial x^i} \frac{\partial x^i}{\partial t^\alpha} + \frac{\partial L_\beta}{\partial x^i_\gamma} \frac{\partial x^i_\gamma}{\partial t^\alpha} \right) dt^\alpha \\
&= \left(\frac{\partial L_\alpha}{\partial t^\beta} + \frac{\partial L_\alpha}{\partial x^i} \frac{\partial x^i}{\partial t^\beta} + \frac{\partial L_\alpha}{\partial x^i_\gamma} \frac{\partial x^i_\gamma}{\partial t^\beta} \right) dt^\alpha \\
&= \frac{\partial L_\alpha}{\partial t^\beta} dt^\alpha + \left(\frac{\partial L_\alpha}{\partial x^i} - D_\gamma \frac{\partial L_\alpha}{\partial x^i_\gamma} \right) \frac{\partial x^i}{\partial t^\beta} dt^\alpha + D_\gamma \left(\frac{\partial x^i}{\partial t^\beta} \frac{\partial L_\alpha}{\partial x^i_\gamma} \right) dt^\alpha \\
&= \left(\frac{\partial L_\alpha}{\partial t^\beta} + D_\gamma \left(\frac{\partial x^i}{\partial t^\beta} \frac{\partial L_\alpha}{\partial x^i_\gamma} \right) \right) dt^\alpha.
\end{aligned}$$

□

Proposition 9.2. *If* $L_\alpha dt^\alpha$ *is a Lagrange* 1*-form which satisfies the multi-time anti-trace Euler-Lagrange-like equations*

$$\frac{\partial L_\alpha}{\partial x^i}\delta^\gamma_\beta - D_\beta \frac{\partial L_\alpha}{\partial x^i_\gamma} = 0,$$

then the Lagrange 1*-forms*

$$\left(\frac{\partial L_\alpha}{\partial t^\beta} + D_\beta\left(\frac{\partial x^i}{\partial t^\gamma}\frac{\partial L_\alpha}{\partial x^i_\gamma}\right)\right) dt^\alpha$$

are exact.

Proof. By direct calculation, we find

$$\begin{aligned}
dL_\beta(t,x(t),x_\gamma(t)) &= \left(\frac{\partial L_\beta}{\partial t^\alpha} + \frac{\partial L_\beta}{\partial x^i}\frac{\partial x^i}{\partial t^\alpha} + \frac{\partial L_\beta}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\alpha}\right) dt^\alpha \\
&= \left(\frac{\partial L_\alpha}{\partial t^\beta} + \frac{\partial L_\alpha}{\partial x^i}\frac{\partial x^i}{\partial t^\beta} + \frac{\partial L_\alpha}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\beta}\right) dt^\alpha \\
&= \frac{\partial L_\alpha}{\partial t^\beta}dt^\alpha + \left(\frac{\partial L_\alpha}{\partial x^i}\frac{\partial x^i}{\partial t^\beta} - x^i_\gamma D_\beta \frac{\partial L_\alpha}{\partial x^i_\gamma}\right) dt^\alpha + D_\beta\left(\frac{\partial x^i}{\partial t^\gamma}\frac{\partial L_\alpha}{\partial x^i_\gamma}\right) dt^\alpha \\
&= \frac{\partial L_\alpha}{\partial t^\beta}dt^\alpha + x^i_\gamma\left(\frac{\partial L_\alpha}{\partial x^i}\delta^\gamma_\beta - D_\beta \frac{\partial L_\alpha}{\partial x^i_\gamma}\right) dt^\alpha + D_\beta\left(\frac{\partial x^i}{\partial t^\gamma}\frac{\partial L_\alpha}{\partial x^i_\gamma}\right) dt^\alpha \\
&= \left(\frac{\partial L_\alpha}{\partial t^\beta} + D_\beta\left(\frac{\partial x^i}{\partial t^\gamma}\frac{\partial L_\alpha}{\partial x^i_\gamma}\right)\right) dt^\alpha.
\end{aligned}$$

□

Open problem. What one can obtain from the relation

$$D_\gamma\left(x^i\left(\frac{\partial L_\alpha}{\partial x^i}\delta^\gamma_\beta - D_\beta\frac{\partial L_\alpha}{\partial x^i_\gamma}\right)\right) = x^i_\gamma\left(\frac{\partial L_\alpha}{\partial x^i}\delta^\gamma_\beta - D_\beta\frac{\partial L_\alpha}{\partial x^i_\gamma}\right) + x^i D_\beta\left(\frac{\partial L_\alpha}{\partial x^i} - D_\gamma\frac{\partial L_\alpha}{\partial x^i_\gamma}\right)?$$

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