

CHEN-RICCI INEQUALITY FOR SUBMANIFOLDS OF CONTACT METRIC MANIFOLDS

MUKUT MANI TRIPATHI

Dedicated to the 65th birthday of Professor Bang-Yen Chen

ABSTRACT. A basic inequality for submanifolds of a Riemannian manifold, which involves Ricci curvature and squared mean curvature of the submanifold, is considered and is named as Chen-Ricci inequality. The conditions under which the inequality becomes Chen-Ricci equality are discussed. The Chen-Ricci inequality for C -totally real submanifolds in (κ, μ) -space forms and non-Sasakian (κ, μ) -manifolds are obtained. In particular, Chen-Ricci inequality for C -totally real submanifolds in Sasakian space forms is derived. Examples of C -totally real submanifolds of Sasakian space forms and non-Sasakian (κ, μ) -manifolds, which satisfy Chen-Ricci equality, are presented. The Chen-Ricci inequalities for submanifolds tangent to the structure vector field in (κ, μ) -space forms and non-Sasakian (κ, μ) -manifolds are obtained. It is shown that invariant submanifolds of non-Sasakian (κ, μ) -manifolds and non-Sasakian (κ, μ) -space forms always satisfy Chen-Ricci equality. An obstruction for an invariant submanifold of a non-Sasakian (κ, μ) -space form $\tilde{M}(c)$ to be Einstein is obtained. In particular, it is deduced that invariant submanifolds of the tangent sphere bundle of a Riemannian manifold of constant curvature $c \neq 1$ and having constant φ -sectional curvature can not be Einstein. The Chen-Ricci inequality for anti-invariant submanifolds tangent to the structure vector field in non-Sasakian (κ, μ) -manifolds are also found. Contrary to a known result that there is no anti-invariant submanifold of a Sasakian space form tangent to the structure vector field, which satisfies the corresponding Chen-Ricci equality; an example of anti-invariant submanifold tangent to the structure vector field in a non-Sasakian (κ, μ) -manifold with $\kappa = 0 = \mu$ is presented, which satisfies Chen-Ricci equality. In last, basic inequalities involving scalar curvature and squared mean curvature for C -totally real submanifolds and submanifolds tangent to the structure vector field of (κ, μ) -space forms and non-Sasakian (κ, μ) -manifolds.

Received: December 23, 2008.

2000 Mathematics Subject Classification: 53C40, 53C15, 53C25, 53D10.

Key words and phrases: Contact metric manifold, Sasakian manifold, (κ, μ) -manifold, (κ, μ) -space form, C -totally real submanifold, anti-invariant submanifold, invariant submanifold, totally umbilical submanifold, totally geodesic submanifold, minimal submanifold, Ricci curvature, k -Ricci curvature, scalar curvature.

1. INTRODUCTION

It is well known that Nash's immersion theorem [37] takes guarantee of an isometric immersion of every n -dimensional Riemannian manifold into the Euclidean space $\mathbb{E}^{n(n+1)(3n+11)/2}$, and this phenomenon provides a natural motivation for the study of submanifolds of Riemannian manifolds. To find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold is one of the basic interests in the submanifold theory. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants namely the Ricci curvature and the scalar curvature. There are also other important modern intrinsic invariants of (sub)manifolds introduced by B.-Y. Chen [16]. For a unit vector X in an n -dimensional submanifold M of a real space form $R^m(c)$, B.-Y. Chen [14, Theorem 4] obtained the following basic inequality

$$\|H\|^2 \geq \frac{4}{n^2} \{\text{Ric}(X) - (n-1)c\} \quad (1.1)$$

involving the Ricci curvature Ric and the squared mean curvature $\|H\|^2$ of the submanifold. The inequality (1.1) drew attention of several authors and they established same kind of inequalities for different kind of submanifolds in ambient manifolds possessing different kind of structures. The submanifolds included mainly invariant, anti-invariant and slant submanifolds [12], while ambient manifolds included mainly real, complex and Sasakian space forms (for example, see [15], [18], [25], [28], [29], [31], [32], [33], [34], [35], [46], [48], [50]). In search of a general theory, it is natural to ask about a basic inequality (corresponding to the inequality (1.1)) involving the Ricci curvature and the squared mean curvature of any submanifold of any Riemannian manifold without assuming any restriction on the Riemann curvature tensor of the ambient manifold. This goal was achieved in [22] (see Theorem 3.1) by exploiting the typical concept of k -Ricci curvature ([14], [52]) and the method of [14, Theorem 4].

On the other hand, the roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations (for more details see [1], [2], [6] and [19]). In fact, the theory of contact metric structures occupies one of the leading places in researches of modern differential geometry because of its several applications in physics such as mechanics, optics, phase space of a dynamical system, thermodynamics and control theory ([1], [2]) and in the theory of geometrical quantization [24]. Furthermore, the internal contents of the theory of contact metric structures are very rich [6] and have close substantial interactions with other parts of geometry.

It is well known [40] that the tangent sphere bundle $T_1\tilde{M}$ of a Riemannian manifold \tilde{M} admits a contact metric structure $(\varphi, \xi, \eta, \tilde{g})$. If \tilde{M} is of constant curvature $c = 1$ then $T_1\tilde{M}$ is Sasakian [45], that is, its curvature tensor \tilde{R} satisfies

$$\tilde{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad X, Y \in T\tilde{M}. \quad (1.2)$$

If $c = 0$ then the curvature tensor \tilde{R} satisfies [5]

$$\tilde{R}(X, Y)\xi = 0. \quad (1.3)$$

As a generalization of these two cases, in [7], D. E. Blair, T. Koufogiorgos and B. J. Papantoniou initiated the study of the class of contact metric manifolds \tilde{M} , which satisfy

$$\tilde{R}(X, Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y), \quad X, Y \in T\tilde{M}, \quad (1.4)$$

where κ, μ are real constants and the $(1, 1)$ -tensor field h is defined by half of the Lie derivative of φ in the characteristic direction ξ . A contact metric manifold belonging to this class is called a (κ, μ) -manifold. Such a structure was first obtained by T. Koufogiorgos [26] by applying a D -homothetic deformation on a contact metric manifold satisfying (1.3), where for a positive constant a , by a D -homothetic deformation [43] one means a change of structure tensors of the form

$$\varphi' = \varphi, \quad \xi' = \frac{1}{a}\xi, \quad \eta' = a\eta, \quad \tilde{g}' = a\tilde{g} + a(a-1)\eta \otimes \eta.$$

There is a complete classification of (κ, μ) -manifolds given by E. Boeckx [10]. In non-Sasakian (κ, μ) -manifolds the condition (1.4) determines the curvature completely; moreover, while the values of κ and μ change, the form of (1.4) is invariant under D -homothetic deformations [7]. Any non-Sasakian (κ, μ) -manifold is locally homogeneous and strongly locally φ -symmetric ([9], [11]). Characteristic examples of non-Sasakian (κ, μ) -manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one and certain Lie groups [10]. If a (κ, μ) -manifold has constant φ -sectional curvature c then it is called a (κ, μ) -space form.

Thus motivated sufficiently, we study Ricci curvature and scalar curvature of submanifolds in contact metric manifolds. To be more specific, the results obtained (in this paper) for submanifolds of (κ, μ) -space forms generalize the corresponding results for submanifolds of Sasakian space forms and at the same time several results and examples obtained for submanifolds of non-Sasakian (κ, μ) -manifolds and non-Sasakian (κ, μ) -space forms are quite different in nature with the corresponding results for submanifolds of Sasakian space forms. The paper is organized as follows. Section 2 consists of a brief introduction to (κ, μ) -manifolds, (κ, μ) -space forms and non-Sasakian (κ, μ) -manifolds. In section 3, we recall the notion of Ricci curvature, k -Ricci curvature, scalar curvature, normalized scalar curvature and give a brief account of submanifolds. Then we recall the result including a basic inequality for submanifolds of any Riemannian manifold, which involves Ricci curvature and squared mean curvature of the submanifold. We also give a simple proof and discuss the conditions under which the inequality becomes equality. Since, this kind of inequality for Ricci curvature of a submanifold in a real space form was first obtained by B.-Y. Chen [14, Theorem 4], we name this inequality as Chen-Ricci inequality. In section 4, we obtain Chen-Ricci inequalities for C -totally real submanifolds in (κ, μ) -space forms and non-Sasakian (κ, μ) -manifolds. In particular,

we derive Chen-Ricci inequality for C -totally real submanifolds in Sasakian space forms. We also give examples of C -totally real submanifolds of Sasakian space forms and non-Sasakian (κ, μ) -manifolds, which satisfy Chen-Ricci equality. In section 5, we establish Chen-Ricci inequalities for submanifolds tangent to the structure vector field in (κ, μ) -space forms and non-Sasakian (κ, μ) -manifolds. As applications of these inequalities, using a surprising fact that every invariant submanifold of a non-Sasakian (κ, μ) -manifold is totally geodesic, we observe that invariant submanifolds of non-Sasakian (κ, μ) -manifolds and non-Sasakian (κ, μ) -space forms satisfy Chen-Ricci equality, which is an improvement of a result of [47], where the Chen-Ricci equality is obtained as an inequality and necessary and sufficient condition for equality case is obtained. In [23] it is proved that a totally geodesic invariant submanifold of a Sasakian space form $\tilde{M}(c)$ is Einstein if and only if $c = 1$. Here, we find an obstruction for an n -dimensional invariant submanifold of a non-Sasakian (κ, μ) -space form $\tilde{M}(c)$ to be Einstein. In particular, we deduce that invariant submanifolds of the tangent sphere bundle of a Riemannian manifold of constant curvature $c \neq 1$ and having constant φ -sectional curvature can not be Einstein. Next, we find Chen-Ricci inequality for anti-invariant submanifolds tangent to the structure vector field in non-Sasakian (κ, μ) -manifolds. Contrary to a known result that there is no anti-invariant submanifold of a Sasakian space form tangent to the structure vector field, which satisfies the corresponding Chen-Ricci equality; we provide an example of anti-invariant submanifold tangent to the structure vector field in a non-Sasakian (κ, μ) -manifold with $\kappa = 0 = \mu$, which satisfies Chen-Ricci equality. In section 6, we find basic inequalities involving scalar curvature and squared mean curvature for C -totally real submanifolds and submanifolds tangent to the structure vector field ξ of (κ, μ) -space forms and non-Sasakian (κ, μ) -manifolds.

2. (κ, μ) -MANIFOLDS

A 1-form η on a differentiable manifold \tilde{M} of odd dimension $2m + 1$ ($m \geq 1$) is called a *contact form* if $\eta \wedge (d\eta)^m \neq 0$ everywhere on \tilde{M} , and \tilde{M} equipped with a contact form is a *contact manifold*. Since rank of $d\eta$ is $2m$ on the Grassmann algebra $\wedge T_p^* \tilde{M}$ at each point $p \in \tilde{M}$, therefore there exists a unique global vector field ξ , called the *characteristic vector field*, such that $\eta(\xi) = 1$, $d\eta(\xi, \cdot) = 0$, and consequently $\mathfrak{L}_\xi \eta = 0$, $\mathfrak{L}_\xi d\eta = 0$, where \mathfrak{L}_ξ denotes the Lie differentiation by ξ . In 1953, S. S. Chern [17] proved that the structural group of a $(2m + 1)$ -dimensional contact manifold can be reduced to $\mathcal{U}(m) \times 1$. A $(2m + 1)$ -dimensional differentiable manifold \tilde{M} is called an *almost contact manifold* [20] if its structural group can be reduced to $\mathcal{U}(m) \times 1$. Equivalently, there is an *almost contact structure* (φ, ξ, η) [39] consisting of a tensor field φ of type $(1, 1)$, a vector field ξ , and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \quad (2.1)$$

First and one of the remaining three relations of (2.1) imply the other two relations of (2.1). An almost contact structure is *normal* [41] if the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$,

where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ , vanishes identically. Let \tilde{g} be a compatible Riemannian metric with (φ, ξ, η) , that is,

$$\langle X, Y \rangle = \langle \varphi X, \varphi Y \rangle + \eta(X)\eta(Y), \quad X, Y \in T\tilde{M}, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of the metric \tilde{g} . Then, \tilde{M} becomes an *almost contact metric manifold* equipped with an *almost contact metric structure* $(\varphi, \xi, \eta, \tilde{g})$. The equation (2.2) is equivalent to

$$\langle X, \varphi Y \rangle = -\langle \varphi X, Y \rangle \quad \text{alongwith} \quad \langle X, \xi \rangle = \eta(X). \tag{2.3}$$

An almost contact metric structure becomes a contact metric structure if $\langle X, \varphi Y \rangle = d\eta(X, Y)$ for all $X, Y \in T\tilde{M}$. In a contact metric manifold \tilde{M} , the $(1, 1)$ -tensor field h defined by $2h = \mathfrak{L}_\xi\varphi$, which is the Lie derivative of φ in the characteristic direction ξ , is symmetric and satisfies

$$h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \tilde{\nabla}\xi = -\varphi - \varphi h, \quad \text{trace}(h) = \text{trace}(\varphi h) = 0. \tag{2.4}$$

A contact metric manifold is called a *K-contact manifold* if the characteristic vector field ξ is a Killing vector field. An almost contact metric manifold is a *K-contact manifold* if and only if $\tilde{\nabla}\xi = -\varphi$, where $\tilde{\nabla}$ is the Levi-Civita connection. A *K-contact manifold* is a contact metric manifold, while the converse is true if $h = 0$. In a *K-contact manifold*, the curvature tensor \tilde{R} satisfies $(\tilde{\nabla}_X\varphi)Y = \tilde{R}(\xi, X)Y$ for all $X, Y \in T\tilde{M}$. A normal contact metric manifold is a *Sasakian manifold*. An almost contact metric manifold is Sasakian if and only if [42]

$$(\tilde{\nabla}_X\varphi)Y = \langle X, Y \rangle \xi - \eta(Y)X, \quad X, Y \in T\tilde{M}. \tag{2.5}$$

Also, a contact metric manifold \tilde{M} is Sasakian if and only if the curvature tensor \tilde{R} satisfies the equation (1.2). A Sasakian manifold is always a *K-contact manifold* and the converse is true in the dimension three. Thus a 3-dimensional contact metric manifold is Sasakian if and only if $h = 0$.

A contact metric manifold is a (κ, μ) -manifold \tilde{M} if it satisfies (1.4). In a (κ, μ) -manifold \tilde{M} , one has [7]

$$(\tilde{\nabla}_X\varphi)Y = \langle X + hX, Y \rangle \xi - \eta(Y)(X + hX), \tag{2.6}$$

$$\begin{aligned} (\nabla_X h)Y &= \{(1 - \kappa)g(X, \varphi Y) - g(X, \varphi hY)\} \xi \\ &\quad + \eta(Y)\{h(\varphi X + \varphi hX)\} - \mu\eta(X)\varphi hY \end{aligned} \tag{2.7}$$

for all $X, Y \in T\tilde{M}$. The Ricci operator \tilde{Q} satisfies $\tilde{Q}\xi = 2m\kappa\xi$, where $\dim(\tilde{M}) = 2m + 1$. Moreover, $h^2 = (\kappa - 1)\varphi^2$ and the eigenvalues of h are 0, λ and $-\lambda$, where $\lambda = \sqrt{1 - \kappa}$. The eigenspace relative to the eigenvalue 0 is $\{\xi\}$. Moreover, for $\kappa \neq 1$, the subbundle $\mathcal{D} = \ker(\eta)$ can be decomposed in the eigenspace distributions \mathcal{D}_+ and \mathcal{D}_- relative to the eigenvalues λ and $-\lambda$, respectively. These distributions are orthogonal to each other and have dimension m . In a (κ, μ) -manifold it follows that $\kappa \leq 1$. In fact, for a (κ, μ) -manifold, the conditions of being a Sasakian manifold, a *K-contact manifold*, $\kappa = 1$ and $h = 0$ are all equivalent. If $\mu = 0$, then a (κ, μ) -manifold is called an *N(κ)-contact metric manifold* [8]. The tangent sphere bundle

$T_1\tilde{M}$ of a Riemannian manifold \tilde{M} of constant curvature c is a (κ, μ) -manifold with $\kappa = c(2 - c)$ and $\mu = -2c$.

Like complex space forms in Hermitian geometry, in contact metric geometry we have the notion of manifolds with constant φ -sectional curvature. In an almost contact metric manifold, for a unit vector X orthogonal to ξ , the sectional curvature $\tilde{K}(X \wedge \varphi X)$ is called a φ -sectional curvature. In [27], T. Koufogiorgos showed that in a (κ, μ) -manifold $(\tilde{M}, \varphi, \xi, \eta, g)$ of dimension > 3 if the φ -sectional curvature at a point p is independent of the φ -section at p , then it is constant. If the (κ, μ) -manifold \tilde{M} has constant φ -sectional curvature c then it is called a (κ, μ) -space form and is denoted by $\tilde{M}(c)$. The Riemann curvature tensor \tilde{R} of $\tilde{M}(c)$ is given by [27]

$$\begin{aligned} \tilde{R}(X, Y)Z &= \frac{c+3}{4} \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \} \\ &+ \frac{c-1}{4} \{ 2 \langle X, \varphi Y \rangle \varphi Z + \langle X, \varphi Z \rangle \varphi Y - \langle Y, \varphi Z \rangle \varphi X \} \\ &+ \frac{c+3-4\kappa}{4} \{ \eta(X) \eta(Z) Y - \eta(Y) \eta(Z) X \\ &\quad + \langle X, Z \rangle \eta(Y) \xi - \langle Y, Z \rangle \eta(X) \xi \} \\ &+ \frac{1}{2} \{ \langle hY, Z \rangle hX - \langle hX, Z \rangle hY \\ &\quad + \langle \varphi hX, Z \rangle \varphi hY - \langle \varphi hY, Z \rangle \varphi hX \} \\ &+ \langle \varphi Y, \varphi Z \rangle hX - \langle \varphi X, \varphi Z \rangle hY \\ &+ \langle hX, Z \rangle \varphi^2 Y - \langle hY, Z \rangle \varphi^2 X \\ &+ \mu \{ \eta(Y) \eta(Z) hX - \eta(X) \eta(Z) hY \\ &\quad + \langle hY, Z \rangle \eta(X) \xi - \langle hX, Z \rangle \eta(Y) \xi \} \end{aligned} \quad (2.8)$$

for all $X, Y, Z \in T\tilde{M}$. For a non-Sasakian (κ, μ) -manifold \tilde{M} , its Riemann curvature tensor \tilde{R} is given by ([9],[10])

$$\begin{aligned} \tilde{R}(X, Y)Z &= \left(1 - \frac{\mu}{2}\right) \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \} \\ &- \frac{\mu}{2} \{ 2 \langle X, \varphi Y \rangle \varphi Z + \langle X, \varphi Z \rangle \varphi Y - \langle Y, \varphi Z \rangle \varphi X \} \\ &+ \langle Y, Z \rangle hX - \langle X, Z \rangle hY - \langle hX, Z \rangle Y + \langle hY, Z \rangle X \\ &+ \frac{1 - (\mu/2)}{1 - \kappa} \{ \langle hY, Z \rangle hX - \langle hX, Z \rangle hY \} \\ &+ \frac{\kappa - (\mu/2)}{1 - \kappa} \{ \langle \varphi hY, Z \rangle \varphi hX - \langle \varphi hX, Z \rangle \varphi hY \} \\ &+ \{ (\kappa - 1 + (\mu/2)) \langle Y, Z \rangle + (\mu - 1) \langle hY, Z \rangle \} \eta(X) \xi \\ &- \eta(X) \eta(Z) \{ (\kappa - 1 + (\mu/2)) Y + (\mu - 1) hY \} \\ &+ \eta(Y) \eta(Z) \{ (\kappa - 1 + (\mu/2)) X + (\mu - 1) hX \} \\ &- \{ (\kappa - 1 + (\mu/2)) \langle X, Z \rangle + (\mu - 1) \langle hX, Z \rangle \} \eta(Y) \xi \end{aligned} \quad (2.9)$$

for all vector fields X, Y, Z in \tilde{M} .

If $\kappa = 1$ then a (κ, μ) -space form $\tilde{M}(c)$ reduces to a Sasakian space form $\tilde{M}(c)$ [38] and (2.8) reduces to

$$\begin{aligned} \tilde{R}(X, Y)Z &= \frac{c+3}{4} \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \} \\ &\quad + \frac{c-1}{4} \{ 2 \langle X, \varphi Y \rangle \varphi Z + \langle X, \varphi Z \rangle \varphi Y - \langle Y, \varphi Z \rangle \varphi X \\ &\quad \quad + \eta(X) \eta(Z) Y - \eta(Y) \eta(Z) X \\ &\quad \quad + \langle X, Z \rangle \eta(Y) \xi - \langle Y, Z \rangle \eta(X) \xi \}. \end{aligned} \tag{2.10}$$

In [44], S. Tanno asked whether there exists a non-Sasakian contact metric manifold of constant φ -sectional curvature. In [27], T. Koufogiorgos answered this problem in affirmative. A 3-dimensional non-Sasakian (κ, μ) -manifold has a constant φ -sectional curvature, but for higher dimension this is not in general true. A non-Sasakian (κ, μ) -manifold is of constant φ -sectional curvature c if and only if $\mu = \kappa + 1$; in this case $c = -2k - 1$. In particular, the tangent sphere bundle of a manifold of constant curvature $c \neq 1$ has constant φ -sectional curvature $c^2 = 9 \pm 4\sqrt{5}$ if and only if $c = 2 \pm \sqrt{5}$. For more details we refer to [6], [7] and [27].

3. CHEN-RICCI INEQUALITY FOR RIEMANNIAN SUBMANIFOLDS

Let M be an n -dimensional Riemannian manifold. Let $\{e_1, \dots, e_k\}$, $2 \leq k \leq n$, be an orthonormal basis of a k -plane section Π_k of $T_p M$. If $k = n$ then $\Pi_n = T_p M$; and if $k = 2$ then Π_2 is a plane section of $T_p M$. For a fixed $i \in \{1, \dots, k\}$, a k -Ricci curvature of Π_k at e_i , denoted $\text{Ric}_{\Pi_k}(e_i)$, is defined by [14]

$$\text{Ric}_{\Pi_k}(e_i) = \sum_{j \neq i}^k K(e_i \wedge e_j), \tag{3.1}$$

where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j . An n -Ricci curvature $\text{Ric}_{T_p M}(e_i)$ is the usual Ricci curvature of e_i , denoted $\text{Ric}(e_i)$. Thus for any orthonormal basis $\{e_1, \dots, e_n\}$ for $T_p M$ and for a fixed $i \in \{1, \dots, n\}$, we have

$$\text{Ric}_{T_p M}(e_i) \equiv \text{Ric}(e_i) = \sum_{j \neq i}^n K(e_i \wedge e_j).$$

The scalar curvature $\tau(\Pi_k)$ of the k -plane section Π_k is given by

$$\tau(\Pi_k) = \sum_{1 \leq i < j \leq k} K(e_i \wedge e_j). \tag{3.2}$$

Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_p(\Pi_k)$ of Π_k at p under the exponential map at p . We define the normalized scalar curvature $\tau_N(\Pi_k)$ of Π_k by [23]

$$\tau_N(\Pi_k) = \frac{2\tau(\Pi_k)}{k(k-1)}. \tag{3.3}$$

The *normalized scalar curvature* at p is defined as [13]

$$\tau_N(p) = \frac{2\tau(p)}{n(n-1)}. \quad (3.4)$$

Then, we see that $\tau_N(p) = \tau_N(T_pM)$. The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p , that is, $\tau(p) = \tau(T_pM)$. If Π_2 is a plane section and $\{e_1, e_2\}$ is any orthonormal basis for Π_2 , then

$$\text{Ric}_{\Pi_2}(e_1) = \text{Ric}_{\Pi_2}(e_2) = \tau(\Pi_2) = \tau_N(\Pi_2) = K_{12}.$$

Let M be a submanifold of a Riemannian manifold \tilde{M} equipped with a Riemannian metric \tilde{g} . Covariant derivatives and curvatures with respect to (M, g) will be written in the usual manner, while those with respect to the ambient manifold (\tilde{M}, \tilde{g}) will be written with “tildes” over them. We use the inner product notation $\langle \cdot, \cdot \rangle$ for both the metrics \tilde{g} of \tilde{M} and the induced metric g on the submanifold M . The Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \quad \text{and} \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N$$

for all $X, Y \in TM$ and $N \in T^\perp M$, where $\tilde{\nabla}$, ∇ and ∇^\perp are respectively the Riemannian, induced Riemannian and induced normal connections in \tilde{M} , M and the normal bundle $T^\perp M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $\langle \sigma(X, Y), N \rangle = \langle A_N X, Y \rangle$. The equation of Gauss is given by

$$R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + \langle \sigma(X, W), \sigma(Y, Z) \rangle - \langle \sigma(X, Z), \sigma(Y, W) \rangle \quad (3.5)$$

for all $X, Y, Z, W \in TM$, where \tilde{R} and R are the curvature tensors of \tilde{M} and M respectively.

The mean curvature vector H is given by $nH = \text{trace}(\sigma)$, where $n = \dim(M)$. The submanifold M is *totally geodesic* in \tilde{M} if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X, Y) = g(X, Y)H$ for all $X, Y \in TM$, then M is *totally umbilical*.

Now, let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r belongs to an orthonormal basis $\{e_{n+1}, \dots, e_m\}$ of the normal space $T_p^\perp M$. We put

$$\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r \rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^n \langle \sigma(e_i, e_j), \sigma(e_i, e_j) \rangle.$$

Let $K(e_i \wedge e_j)$ and $\tilde{K}(e_i \wedge e_j)$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold \tilde{M} respectively. In view of the equation (3.5) of Gauss, we have

$$K(e_i \wedge e_j) = \tilde{K}(e_i \wedge e_j) + \sum_{r=n+1}^m (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2). \quad (3.6)$$

From (3.6) it follows that

$$2\tau(p) = 2\tilde{\tau}(T_pM) + n^2 \|H\|^2 - \|\sigma\|^2, \tag{3.7}$$

where

$$\tilde{\tau}(T_pM) = \sum_{1 \leq i < j \leq n} \tilde{K}(e_i \wedge e_j)$$

denote the scalar curvature of the n -plane section T_pM in the ambient manifold \tilde{M} . Also, the squared second fundamental form and the squared mean curvature satisfy

$$\begin{aligned} \|\sigma\|^2 &= \frac{1}{2}n^2\|H\|^2 + \frac{1}{2} \sum_{r=n+1}^m (\sigma_{11}^r - \sigma_{22}^r - \dots - \sigma_{nn}^r)^2 \\ &+ 2 \sum_{r=n+1}^m \sum_{j=2}^n (\sigma_{1j}^r)^2 - 2 \sum_{r=n+1}^m \sum_{2 \leq i < j \leq n} (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2). \end{aligned} \tag{3.8}$$

The *relative null space* of M at p is defined by [14]

$$\mathcal{N}_p = \{X \in T_pM \mid \sigma(X, Y) = 0 \text{ for all } Y \in T_pM\},$$

which is also known as the *kernel of the second fundamental form* at p [15]. We denote the set of unit vectors in T_pM by T_p^1M ; thus

$$T_p^1M = \{X \in T_pM \mid \langle X, X \rangle = 1\}.$$

Now, we recall the following result, as a general theory, in which a basic inequality involving the Ricci curvature and the squared mean curvature of any submanifold of a Riemannian manifold is established. From now on, the inequality (3.9) will be called the *Chen-Ricci inequality*. Equality case of Chen-Ricci inequality will be known as the *Chen-Ricci equality*.

Theorem 3.1. ([22, Theorem 3.1]) *Let M be an n -dimensional submanifold of a Riemannian manifold. Then, the following statements are true.*

(a) *For $X \in T_p^1M$, it follows that*

$$\text{Ric}(X) \leq \frac{1}{4}n^2\|H\|^2 + \widetilde{\text{Ric}}_{(T_pM)}(X), \tag{3.9}$$

where $\widetilde{\text{Ric}}_{(T_pM)}(X)$ is the n -Ricci curvature of T_pM at X with respect to the ambient manifold \tilde{M} .

(b) *The equality case of (3.9) is satisfied by $X \in T_p^1M$ if and only if*

$$\sigma(X, X) = \frac{n}{2}H(p) \quad \text{and} \quad \sigma(X, Y) = 0 \tag{3.10}$$

for all $Y \in T_pM$ such that $\langle X, Y \rangle = 0$.

(c) *The equality case of (3.9) holds for all $X \in T_p^1M$ if and only if either p is a totally geodesic point or $n = 2$ and p is an umbilical point.*

Proof. We give a very simple proof taken from [21]. We put

$$2\sigma'(X, Y) = \sigma(X, Y) - \frac{n}{2} \langle X, Y \rangle H, \quad X, Y \in T_p M.$$

Then for a unit vector $X \in T_p M$, we obtain

$$0 \leq \sum_{i=1}^n \langle \sigma'(X, e_i), \sigma'(X, e_i) \rangle = \sum_{i=1}^n \langle \sigma(X, e_i), \sigma(X, e_i) \rangle - n \langle H, \sigma(X, X) \rangle + \frac{n^2}{4} \|H\|^2.$$

In view of the Gauss equation (3.5) and the above inequality, we can easily get our theorem. \square

From Theorem 3.1, we immediately have the following

Corollary 3.1. *Let M be an n -dimensional submanifold of a Riemannian manifold. Then for any $X \in T_p^1 M$ any two of the following three statements imply the remaining one.*

- (a) *The mean curvature vector $H(p)$ vanishes.*
- (b) *The unit vector X belongs to the relative null space \mathcal{N}_p .*
- (c) *The unit vector X satisfies the equality case of (3.9), namely*

$$\text{Ric}(X) = \frac{1}{4} n^2 \|H(p)\|^2 + \widetilde{\text{Ric}}_{(T_p M)}(X). \quad (3.11)$$

4. CHEN-RICCI INEQUALITY FOR C -TOTALLY REAL SUBMANIFOLDS

A submanifold M of an almost contact metric manifold is said to be *anti-invariant* ([54], [55]) if $\varphi(TM) \subseteq T^\perp M$. A submanifold M in a contact manifold is called an *integral submanifold* [6] if every tangent vector of M belongs to the contact distribution defined by $\eta = 0$. In this case we have $d\eta(X, Y) = 0$, $X, Y \in TM$. An integral submanifold M in a contact metric manifold is called a *C -totally real submanifold* [53]. If a submanifold M in a contact metric manifold is normal to the structure vector field ξ , then it is anti-invariant. Thus C -totally real submanifolds in a contact metric manifold are anti-invariant, as they are normal to ξ .

For a submanifold M of an almost contact metric manifold, we put

$$\|Q\|^2 = \sum_{i,j=1}^n \langle Qe_i, e_j \rangle^2, \quad Q \in \{P, h^T, (\varphi h)^T\},$$

where PX , $h^T X$ and $(\varphi h)^T X$ are the tangential parts of φX , hX and φhX respectively for all $X \in TM$. If M is C -totally real then in view of

$$\langle A_\xi X, Y \rangle = \langle -\tilde{\nabla}_X \xi, Y \rangle = \langle \varphi X + \varphi hX, Y \rangle,$$

it follows that $A_\xi = (\varphi h)^T$.

Let M be a submanifold of a (κ, μ) -manifold \tilde{M} . If \tilde{M} is a non-Sasakian (κ, μ) -manifold, we define the 1-form, denoted by t_2 on M by [47]

$$t_2(X) = \frac{1 - (\mu/2)}{1 - \kappa} \left\{ \|h^T X\|^2 - \text{trace}(h^T) \langle h^T X, X \rangle \right\} + \frac{\kappa - (\mu/2)}{1 - \kappa} \left\{ \|(\varphi h)^T X\|^2 - \text{trace}((\varphi h)^T) \langle (\varphi h)^T X, X \rangle \right\} \quad (4.1)$$

for all $X \in TM$. If \tilde{M} is a (κ, μ) -space form, we write

$$t_2(X) = \frac{1}{2} \left\{ \|h^T X\|^2 - \text{trace}(h^T) \langle h^T X, X \rangle - \|(\varphi h)^T X\|^2 + \text{trace}((\varphi h)^T) \langle (\varphi h)^T X, X \rangle \right\}. \quad (4.2)$$

Now, we obtain Chen-Ricci inequality for a C -totally real submanifold in a (κ, μ) -space form.

Theorem 4.1. *Let M be an n -dimensional C -totally real submanifold of a contact metric manifold \tilde{M} . Then, the following statements are true.*

(a) *If \tilde{M} is a (κ, μ) -space form $\tilde{M}(c)$ then the Chen-Ricci inequality is given by*

$$\text{Ric}(X) \leq \frac{1}{4} n^2 \|H\|^2 + \frac{1}{4} (n - 1) (c + 3) + \text{trace}(h^T) + (n - 2) \langle h^T X, X \rangle - t_2(X) \quad (4.3)$$

for every $X \in T_p^1 M$.

(b) *If \tilde{M} is a non-Sasakian (κ, μ) -manifold then the Chen-Ricci inequality is given by*

$$\text{Ric}(X) \leq \frac{1}{4} n^2 \|H\|^2 + (n - 1) \left(1 - \frac{\mu}{2} \right) + \text{trace}(h^T) + (n - 2) \langle h^T X, X \rangle - t_2(X) \quad (4.4)$$

for every $X \in T_p^1 M$.

(c) *A unit vector $X \in T_p M$ satisfies the equality case of (4.3) (resp. (4.4)) if and only if (3.10) is true. In particular, if $H(p) = 0$, then a unit vector $X \in T_p M$ satisfies the equality case of (4.3) (resp. (4.4)) if and only if X belongs to the relative null space \mathcal{N}_p .*

(d) *The equality case of (4.3) (resp. (4.4)) is true for all unit vectors in $T_p M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space $T_p M$. Then from (2.8) it follows that

$$\begin{aligned} \tilde{K}(e_i \wedge e_j) &= \frac{c + 3}{4} + \langle h^T e_i, e_i \rangle + \langle h^T e_j, e_j \rangle \\ &\quad + \frac{1}{2} \left\{ \langle h^T e_i, e_i \rangle \langle h^T e_j, e_j \rangle - \langle h^T e_i, e_j \rangle^2 \right. \\ &\quad \left. - \langle (\varphi h)^T e_i, e_i \rangle \langle (\varphi h)^T e_j, e_j \rangle + \langle (\varphi h)^T e_i, e_j \rangle^2 \right\}. \end{aligned} \quad (4.5)$$

From (4.5) we have

$$\begin{aligned} \widetilde{\text{Ric}}_{(T_p M)}(e_i) &= \sum_{j \neq i}^n \frac{c+3}{4} + \sum_{j \neq i}^n \langle h^T e_i, e_i \rangle + \sum_{j \neq i}^n \langle h^T e_j, e_j \rangle \\ &\quad + \frac{1}{2} \left\{ \sum_{j \neq i}^n \langle h^T e_i, e_i \rangle \langle h^T e_j, e_j \rangle - \sum_{j \neq i}^n \langle h^T e_i, e_j \rangle^2 \right. \\ &\quad \left. - \sum_{j \neq i}^n \langle (\varphi h)^T e_i, e_i \rangle \langle (\varphi h)^T e_j, e_j \rangle + \sum_{j \neq i}^n \langle (\varphi h)^T e_i, e_j \rangle^2 \right\} \end{aligned}$$

or

$$\begin{aligned} \widetilde{\text{Ric}}_{(T_p M)}(e_i) &= \frac{1}{4} (n-1)(c+3) + (n-1) \langle h^T e_i, e_i \rangle \\ &\quad + (\text{trace}(h^T) - \langle h^T e_i, e_i \rangle) \\ &\quad + \frac{1}{2} \left\{ \langle h^T e_i, e_i \rangle (\text{trace}(h^T) - \langle h^T e_i, e_i \rangle) \right. \\ &\quad \left. - \left(\sum_{j=1}^n \langle h^T e_i, e_j \rangle^2 - \langle h^T e_i, e_i \rangle^2 \right) \right. \\ &\quad \left. - \langle (\varphi h)^T e_i, e_i \rangle (\text{trace}((\varphi h)^T) - \langle (\varphi h)^T e_i, e_i \rangle) \right. \\ &\quad \left. + \left(\sum_{j=1}^n \langle (\varphi h)^T e_i, e_j \rangle^2 - \langle (\varphi h)^T e_i, e_i \rangle^2 \right) \right\}. \end{aligned}$$

Consequently,

$$\widetilde{\text{Ric}}_{(T_p M)}(e_i) = \frac{1}{4} (n-1)(c+3) + \text{trace}(h^T) + (n-2) \langle h^T e_i, e_i \rangle - \mathbf{t}_2(e_i), \quad (4.6)$$

where (4.2) is also used. Using (4.6) in (3.9) we find the Chen-Ricci inequality (4.3).

Similarly, in view of (2.9) it follows that

$$\begin{aligned} \tilde{K}(e_i \wedge e_j) &= \left(1 - \frac{\mu}{2}\right) + \langle h^T e_i, e_i \rangle + \langle h^T e_j, e_j \rangle \\ &\quad + \frac{1 - (\mu/2)}{1 - \kappa} \left\{ \langle h^T e_i, e_i \rangle \langle h^T e_j, e_j \rangle - \langle h^T e_i, e_j \rangle^2 \right\} \\ &\quad + \frac{\kappa - (\mu/2)}{1 - \kappa} \left\{ \langle (\varphi h)^T e_i, e_i \rangle \langle (\varphi h)^T e_j, e_j \rangle - \langle (\varphi h)^T e_i, e_j \rangle^2 \right\}. \quad (4.7) \end{aligned}$$

From (4.7) and (4.1) we get

$$\widetilde{\text{Ric}}_{(T_p M)}(e_i) = (n-1) \left(1 - \frac{\mu}{2}\right) + \text{trace}(h^T) + (n-2) \langle h^T e_i, e_i \rangle - \mathbf{t}_2(e_i). \quad (4.8)$$

Finally the inequality (4.4) is obtained by using (4.8) in (3.9). Rest of the proof is straightforward. \square

Putting $h = 0$ in (4.3), we have the following

Theorem 4.2. *If M is an n -dimensional C -totally real submanifold of a Sasakian space form $\tilde{M}(c)$, then the following statements are true.*

(a) *It follows that*

$$\text{Ric}(X) \leq \frac{1}{4} \{n^2\|H\|^2 + (n - 1)(c + 3)\}, \quad X \in T_p^1M. \tag{4.9}$$

(b) *The equality case of (4.9) is satisfied by $X \in T_p^1M$ if and only if (3.10) is true. If $H(p) = 0$, $X \in T_p^1M$ satisfies equality in (4.9) if and only if $X \in \mathcal{N}_p$.*

(c) *The equality case of (4.9) holds for all $X \in T_p^1M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

By polarization, from Theorem 4.2, we derive

Theorem 4.3. *Let M be an n -dimensional C -totally real submanifold of a Sasakian space form $\tilde{M}(c)$. Then*

$$S \leq \frac{1}{4} \{n^2\|H\|^2 + (n - 1)(c + 3)\}g. \tag{4.10}$$

with the equality case holding identically if and only if either M is totally geodesic submanifold or M is a totally umbilical surface.

Remark 4.1. The Chen-Ricci inequality (4.3) is same as the inequality (33) in [49] but is different from the inequality (3.1) in [3, Theorem 3.1]. Moreover, the assumption of the statement in [3, Corollary 3.2] is also not true, because the submanifold M in a contact metric manifold normal to the structure vector ξ can not be invariant. The inequality (4.9) is the same as the inequalities (5.1) of Theorem 5.1 in [23], (i) of Theorem 1 in [31], (2.1) of Theorem 2.1 in [35] and (2.1) in Theorem 2.1 in [32]. The inequality (4.10) is same as the inequalities (5.3) of Theorem 5.2 in [23], (9) in the Theorem 3.1 in [29] or [28], (2.9) in Theorem 2.2 in [32] and the inequality in Theorem 2 in [31]).

Now, we give examples of C -totally real submanifolds of Sasakian space forms and non-Sasakian (κ, μ) -manifolds, which satisfy Chen-Ricci equality.

Example 4.1. An n -dimensional compact minimal C -totally real submanifold of a Sasakian space form $M^{2n+1}(c)$, $c > -3$ with positive sectional curvature is totally geodesic [51, Theorem 4]; thus it is an Einstein manifold and satisfies $4S = (n - 1)(c + 3)g$.

Example 4.2. Given a point p of a non-Sasakian (κ, μ) -manifold, there are at least two C -totally real submanifolds, passing through p . To see this, consider the foliations given by the eigendistributions of h ; then their leaves are C -totally real submanifolds of the given non-Sasakian (κ, μ) -manifold. These submanifolds are totally geodesic, therefore the equality case of (4.4) is satisfied.

5. CHEN-RICCI INEQUALITY FOR SUBMANIFOLDS TANGENT TO THE STRUCTURE VECTOR FIELD ξ

Let M be a submanifold of a contact metric manifold such that the structure vector field ξ is tangent to M . Then from (2.4) we get

$$0 = -\varphi\xi - \varphi h\xi = \tilde{\nabla}_\xi \xi = \nabla_\xi \xi + \sigma(\xi, \xi),$$

which gives

$$\nabla_\xi \xi = 0 \quad \text{and} \quad \sigma(\xi, \xi) = 0. \quad (5.1)$$

Now, if M is also totally umbilical then $H = \langle \xi, \xi \rangle H = \sigma(\xi, \xi) = 0$ and we have the following

Lemma 5.1. *Every totally umbilical submanifold of a contact metric manifold, such that the structure vector field of the ambient manifold is tangent to the submanifold, is minimal and consequently totally geodesic.*

In the following Theorem, we find Chen-Ricci inequality for submanifolds tangent to the structure vector field in a (κ, μ) -manifold.

Theorem 5.1. *Let M be an n -dimensional submanifold of a contact metric manifold \tilde{M} such that $\xi \in TM$. Then the following statements are true:*

(a) *If \tilde{M} is a (κ, μ) -space form $\tilde{M}(c)$ then the Chen-Ricci inequality is given by*

$$\begin{aligned} \text{Ric}(X) \leq & \frac{1}{4}n^2 \|H\|^2 + \kappa + \frac{1}{4}(n-2)(c+3) + \frac{3}{4}(c-1)\|PX\|^2 \\ & + (\mu+n-3)\langle h^T X, X \rangle + \text{trace}(h^T) - \mathfrak{t}_2(X) \\ & + \left\{ \frac{1}{4}(n-2)(4\kappa - c - 3) + (\mu-1)\text{trace}(h^T) \right\} \eta(X)^2 \end{aligned} \quad (5.2)$$

for all unit vectors $X \in T_p^1 M$.

(b) *If \tilde{M} is a non-Sasakian (κ, μ) -manifold then the Chen-Ricci inequality is given by*

$$\begin{aligned} \text{Ric}(X) \leq & \frac{1}{4}n^2 \|H\|^2 + \kappa + \frac{1}{2}(n-2)(2-\mu) - \frac{3\mu}{2}\|PX\|^2 \\ & + (\mu+n-3)\langle h^T X, X \rangle + \text{trace}(h^T) - \mathfrak{t}_2(X) \\ & + \left\{ (n-2)\left(\kappa - 1 + \frac{\mu}{2}\right) + (\mu-1)\text{trace}(h^T) \right\} \eta(X)^2, \end{aligned} \quad (5.3)$$

where X is any unit vector in $T_p^1 M$.

(c) *A unit vector $X \in T_p M$ satisfies the equality case of (5.2) (resp. (5.3)) if and only if (3.10) is true. In particular, if $H(p) = 0$, then a unit vector $X \in T_p M$ satisfies the equality case of (5.2) (resp. (5.3)) if and only if X belongs to the relative null space \mathcal{N}_p .*

(d) *The equality in (5.2) (resp. (5.3)) holds identically for all unit tangent vectors in $T_p M$ if and only if p is a totally geodesic point.*

Proof. Let M be an n -dimensional submanifold of a (κ, μ) -space form $\tilde{M}(c)$ such that the structure vector field ξ is tangent to M . Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space T_pM . From (2.8) it follows that

$$\begin{aligned} \tilde{K}(e_i \wedge e_j) &= \frac{c+3}{4} + \frac{3(c-1)}{4} \langle Pe_i, e_j \rangle^2 + \langle h^T e_i, e_i \rangle + \langle h^T e_j, e_j \rangle \\ &\quad + \frac{1}{2} \left\{ \langle h^T e_i, e_i \rangle \langle h^T e_j, e_j \rangle - \langle h^T e_i, e_j \rangle^2 \right\} \\ &\quad - \frac{1}{2} \left\{ \langle (\varphi h)^T e_i, e_i \rangle \langle (\varphi h)^T e_j, e_j \rangle - \langle (\varphi h)^T e_i, e_j \rangle^2 \right\} \\ &\quad - \frac{c+3-4\kappa}{4} \left\{ \eta(e_i)^2 + \eta(e_j)^2 \right\} \\ &\quad + (\mu-1) \left\{ \langle h^T e_i, e_i \rangle \eta(e_j)^2 + \langle h^T e_j, e_j \rangle \eta(e_i)^2 \right. \\ &\quad \left. - 2\eta(e_i)\eta(e_j) \langle h^T e_i, e_j \rangle \right\}. \end{aligned} \tag{5.4}$$

From (5.4) we get

$$\begin{aligned} \widetilde{\text{Ric}}_{(T_pM)}(e_i) &= \kappa + \frac{1}{4}(n-2)(c+3) + \frac{3}{4}(c-1)\|Pe_i\|^2 \\ &\quad + (\mu+n-3) \langle h^T e_i, e_i \rangle + \text{trace}(h^T) - \mathbf{t}_2(e_i) \\ &\quad + \left\{ \frac{1}{4}(n-2)(4\kappa-c-3) + (\mu-1)\text{trace}(h^T) \right\} \eta(e_i)^2. \end{aligned} \tag{5.5}$$

Using (5.5) in (3.9) we find the Chen-Ricci inequality (5.2).

From (2.9) it follows that

$$\begin{aligned} \tilde{K}(e_i \wedge e_j) &= 1 - \frac{\mu}{2} - \frac{3\mu}{2} \langle Pe_i, e_j \rangle^2 + \langle h^T e_i, e_i \rangle + \langle h^T e_j, e_j \rangle \\ &\quad + \frac{1-(\mu/2)}{1-\kappa} \left\{ \langle h^T e_i, e_i \rangle \langle h^T e_j, e_j \rangle - \langle h^T e_i, e_j \rangle^2 \right\} \\ &\quad + \frac{\kappa-(\mu/2)}{1-\kappa} \left\{ \langle (\varphi h)^T e_i, e_i \rangle \langle (\varphi h)^T e_j, e_j \rangle - \langle (\varphi h)^T e_i, e_j \rangle^2 \right\} \\ &\quad + \{ \kappa - 1 + (\mu/2) \} \left\{ \eta(e_i)^2 + \eta(e_j)^2 \right\} \\ &\quad + (\mu-1) \left\{ \eta(e_j)^2 \langle h^T e_i, e_i \rangle + \eta(e_i)^2 \langle h^T e_j, e_j \rangle \right. \\ &\quad \left. - 2\eta(e_i)\eta(e_j) \langle h^T e_i, e_j \rangle \right\}, \end{aligned} \tag{5.6}$$

which gives

$$\begin{aligned} \widetilde{\text{Ric}}_{(T_pM)}(e_i) &= \kappa + \frac{1}{2}(n-2)(2-\mu) - \frac{3\mu}{2}\|Pe_i\|^2 \\ &\quad + (\mu+n-3) \langle h^T e_i, e_i \rangle + \text{trace}(h^T) - \mathbf{t}_2(e_i) \\ &\quad + \left\{ (n-2) \left(\kappa - 1 + \frac{\mu}{2} \right) + (\mu-1) \text{trace}(h^T) \right\} \eta(e_i)^2. \end{aligned} \tag{5.7}$$

Using (5.7) in (3.9) we find the Chen-Ricci inequality (5.3). The proof of (c) is straightforward. Since $\xi \in TM$, therefore from Lemma 5.1, each umbilical point is a totally geodesic point; thus (d) is correct. \square

An n -dimensional submanifold M of an almost contact metric manifold \tilde{M} is said to be *invariant* [55] if $\varphi(TM) \subseteq TM$. If \tilde{M} is a contact metric manifold equipped with the contact metric structure $(\varphi, \xi, \eta, \tilde{g})$, then the structure vector field ξ becomes tangent to the invariant submanifold M , $\sigma(X, \xi) = 0$ and M is minimal [6]. Choosing an orthonormal basis $\{e_i, \varphi e_i, \xi\}$, $i = 1, \dots, (n-1)/2$, we get

$$\|P\|^2 = n - 1, \text{ trace}(h^T) = \text{trace}((\varphi h)^T) = 0, \|(\varphi h)^T\| = \|h^T\|. \quad (5.8)$$

In general, an invariant submanifold of a Sasakian manifold needs not to be totally geodesic. For example, the circle bundle (S, Q^n) over an n -dimensional complex quadric Q^n in a complex projective space $\mathbb{C}P^{n+1}$ is an invariant submanifold of a $(2n+3)$ -dimensional Sasakian space form $S^{2n+3}(c)$ with $c > -3$, which is not totally geodesic [55, pp. 328-329]. But, in non-Sasakian case, we have the following useful result.

Theorem 5.2. [36, Theorem 3.1] *Every invariant submanifold of a non-Sasakian (κ, μ) -manifold is totally geodesic.*

In view of Theorems 5.1 and 5.2 we have the following

Theorem 5.3. *For an n -dimensional invariant submanifold M in a contact metric manifold \tilde{M} the following statements are true:*

- (a) *If \tilde{M} is a non-Sasakian (κ, μ) -manifold then the invariant submanifold M satisfies Chen-Ricci equality*

$$\begin{aligned} \text{Ric}(X) &= \kappa + \frac{1}{2}((n-2)(2-\mu) - 3\mu) + (\mu + n - 3) \langle h^T X, X \rangle \\ &\quad - \frac{1}{2}((n-2)(2-\mu - 2\kappa) - 3\mu) \eta(X)^2 \\ &\quad - \frac{1}{1-k}(\kappa + 1 - \mu) \|h^T X\|^2 \end{aligned} \quad (5.9)$$

for each unit vector $X \in T_p M$.

- (b) *If \tilde{M} is a non-Sasakian (κ, μ) -space form $\tilde{M}(c)$ then the invariant submanifold M satisfies Chen-Ricci equality*

$$\begin{aligned} \text{Ric}(X) &= \frac{1}{4}((n+1)c + 3(n-3) + 4\kappa) + (\mu + n - 3) \langle h^T X, X \rangle \\ &\quad - \frac{1}{4}((n+1)c + 3(n-3) - 4(n-2)\kappa) \eta(X)^2. \end{aligned} \quad (5.10)$$

for each unit vector $X \in T_p M$.

Remark 5.1. The part (a) of the above Theorem is an improvement of Theorem 7.4 of [47], where the Chen-Ricci equality (5.9) is obtained as an inequality and necessary and sufficient condition for equality case is given.

In Theorem 4.5 of [23], it is proved that a totally geodesic invariant submanifold of a Sasakian space form $\tilde{M}(c)$ is Einstein if and only if $c = 1$. Here, we prove the following

Proposition 5.1. *Let M be an n -dimensional invariant submanifold of a non-Sasakian (κ, μ) -space form $\tilde{M}(c)$. If M is Einstein, then*

$$c = -\frac{1}{3} \left(5 - \frac{8}{n-1} \right). \tag{5.11}$$

Proof. Let M be an n -dimensional invariant submanifold of a non-Sasakian (κ, μ) -space form $\tilde{M}(c)$. Suppose that M is Einstein. Let X be a unit vector in $T_p^1 M - \{\xi\}$ orthogonal to ξ . Then from (5.10) it follows that

$$0 = \text{Ric}(X) - \text{Ric}(\xi) = (\mu + n - 3) \langle h^T X, X \rangle + \frac{1}{4} ((n + 1)c + 3(n - 3) - 4(n - 2)\kappa).$$

Since $\text{trace}(h^T) = 0$, from the previous equation we get

$$0 = (n + 1)c + 3(n - 3) - 4(n - 2)\kappa.$$

Using $c = -2\kappa - 1$ from the above equation we get (5.11). □

As an important deduction, we have the following

Theorem 5.4. *Invariant submanifolds of the tangent sphere bundle of a Riemannian manifold of constant curvature $c \neq 1$ and having constant φ -sectional curvature can not be Einstein.*

Proof. It is known that the tangent sphere bundle \tilde{M} of a manifold of constant curvature $c \neq 1$ has constant φ -sectional curvature $c^2 = 9 \pm 4\sqrt{5}$ if and only if $c = 2 \pm \sqrt{5}$ [27]. Now, let M be an invariant submanifold of the tangent sphere bundle \tilde{M} . If M is Einstein then in view of Proposition 5.1, we have

$$9 \pm 4\sqrt{5} = -\frac{1}{3} \left(5 - \frac{8}{n-1} \right),$$

which has no integer solution for n . □

Now, we consider the anti-invariant submanifolds of contact metric manifolds tangent to the structure vector field ξ . Putting $P = 0$ in the Chen-Ricci inequality (5.3), we immediately get the following

Corollary 5.1. *Let M be an n -dimensional anti-invariant submanifold in a non-Sasakian (κ, μ) -manifold such that $\xi \in TM$. Then:*

(a) *The Chen-Ricci inequality becomes*

$$\begin{aligned} \text{Ric}(X) \leq & \frac{1}{4} n^2 \|H\|^2 + \kappa + \frac{1}{2} (n - 2) (2 - \mu) \\ & + (\mu + n - 3) \langle h^T X, X \rangle + \text{trace}(h^T) - t_2(X) \\ & + \left\{ (n - 2) \left(\kappa - 1 + \frac{\mu}{2} \right) + (\mu - 1) \text{trace}(h^T) \right\} \eta(X)^2, \end{aligned} \tag{5.12}$$

where X is any unit vector in T_pM .

- (b) The unit vector $X \in T_pM$ satisfies the equality case of (5.12) if and only if (3.10) is true. In particular, if $H(p) = 0$, then the unit vector $X \in T_pM$ satisfies the equality case of (5.12) if and only if X belongs to the relative null space \mathcal{N}_p .
- (c) The equality in (5.12) holds identically for all unit tangent vectors in T_pM if and only if p is a totally geodesic point.

Now, we recall the following result.

Theorem 5.5. (Theorem 4.12 (b), [23]) *Let M be an n -dimensional anti-invariant submanifold of a Sasakian space form $\tilde{M}(c)$ tangent to ξ . If $X \in T_pM$ is a unit vector perpendicular to ξ then*

$$\text{Ric}(X) < \frac{1}{4} \{n^2\|H\|^2 + (n-1)(c+3) - (c-1)\}. \quad (5.13)$$

From the previous result, it is clear that an anti-invariant submanifold of a Sasakian space form tangent to the structure vector field can not satisfy the Chen-Ricci equality for arbitrary unit vectors. In contrast to this fact, we give the following

Example 5.1. Let $(\tilde{M}, \varphi, \xi, \eta, g)$ be a contact metric manifold satisfying $\tilde{R}(X, Y)\xi = 0$ for all $X, Y \in TM$. These manifolds have been deeply studied by D. Blair in [5], where he proves that the distribution $\mathcal{D}_+ \oplus \{\xi\}$ is integrable and defines a totally geodesic foliation of M , where \mathcal{D}_+ is the eigenspace distribution corresponding to the eigenvalue $\lambda = 1$ of h . Thus the leaves of $\mathcal{D}_+ \oplus \{\xi\}$ give examples of totally geodesic submanifolds of \tilde{M} , which are anti-invariant because $\varphi\mathcal{D}_+ = \mathcal{D}_-$ so that φ maps tangent vectors into normal vectors. The unit vectors in these leaves satisfy the Chen-Ricci equality (the equality case of (5.12)), namely

$$\begin{aligned} \text{Ric}(X) &= (n-2) + (\mu + n - 3) \langle h^T X, X \rangle + \text{trace}(h^T) \\ &\quad + \langle h^T X, X \rangle \text{trace}(h^T) - \|h^T X\|^2 \\ &\quad - \{(n-2) + \text{trace}(h^T)\} \eta(X)^2, \end{aligned} \quad (5.14)$$

where $\kappa = \mu = \|H\| = 0$ has been used in (5.12). In particular, $\text{Ric}(\xi) = 0$.

6. SCALAR CURVATURE OF SUBMANIFOLDS

In view of (3.7) it follows that an n -dimensional submanifold M of a Riemannian manifold satisfies

$$\tau(p) \leq \frac{1}{2} n^2 \|H\|^2 + \tilde{\tau}(T_pM) \quad (6.1)$$

at each point $p \in M$ with equality if and only if p is a totally geodesic point. The improved version of the inequality (6.1) is given in the following:

Theorem 6.1. ([21, Theorem 4.2]) *For an n -dimensional submanifold M in an m -dimensional Riemannian manifold, at each point $p \in M$, we have*

$$\tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \tilde{\tau}(T_p M) \tag{6.2}$$

or equivalently

$$\tau_N(p) \leq \|H\|^2 + \tilde{\tau}_N(T_p M) \tag{6.3}$$

with equality if and only if p is a totally umbilical point.

Definition 6.1. The inequality (6.2) or (6.3) will be known as the *scalar inequality*.

Let M be a submanifold of a (κ, μ) -manifold \tilde{M} . If \tilde{M} is a non-Sasakian (κ, μ) -manifold, we write [47]

$$\begin{aligned} \mathfrak{t}_M &= \frac{1 - (\mu/2)}{1 - \kappa} \{(\text{trace}(h^T))^2 - \|h^T\|^2\} \\ &\quad + \frac{\kappa - (\mu/2)}{1 - \kappa} \{(\text{trace}((\varphi h)^T))^2 - \|(\varphi h)^T\|^2\}. \end{aligned} \tag{6.4}$$

If \tilde{M} is a (κ, μ) -space form, we write

$$\mathfrak{t}_M = \frac{1}{2} \{(\text{trace}(h^T))^2 - \|h^T\|^2 - (\text{trace}((\varphi h)^T))^2 + \|(\varphi h)^T\|^2\}. \tag{6.5}$$

Now, we obtain scalar inequality for submanifolds of (κ, μ) -manifolds. In fact, we have the following

Theorem 6.2. *Let M be an n -dimensional C -totally real submanifold of a contact metric manifold \tilde{M} . Then the following statements are true.*

(a) *If \tilde{M} is a (κ, μ) -space form $\tilde{M}(c)$ then the scalar inequality is given by*

$$\tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \frac{1}{8} n(n-1)(c+3) + (n-1) \text{trace}(h^T) + \frac{1}{2} \mathfrak{t}_M. \tag{6.6}$$

(b) *If \tilde{M} is a non-Sasakian (κ, μ) -manifold then the scalar inequality is given by*

$$\tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \frac{1}{2} n(n-1) \left(1 - \frac{\mu}{2}\right) + (n-1) \text{trace}(h^T) + \frac{1}{2} \mathfrak{t}_M. \tag{6.7}$$

(c) *The equality cases of (6.6) (resp. (6.7)) are true if and only if p is a totally umbilical point.*

Proof. From (4.6) and (6.5) we get

$$\tilde{\tau}(T_p M) = \frac{1}{8} n(n-1)(c+3) + (n-1) \text{trace}(h^T) + \frac{1}{2} \mathfrak{t}_M, \tag{6.8}$$

which in view of (6.2) gives (6.6). Similarly, from (4.8) and (6.4) we get

$$\tilde{\tau}(T_p M) = \frac{1}{2} n(n-1) \left(1 - \frac{\mu}{2}\right) + (n-1) \text{trace}(h^T) + \frac{1}{2} \mathfrak{t}_M. \tag{6.9}$$

Using (6.9) in (6.2) gives (6.7). □

Using $h = 0$, we immediately have the following

Corollary 6.1. *Let M be an n -dimensional C -totally real submanifold of a Sasakian space form $\tilde{M}(c)$. Then the scalar inequality is given by*

$$\tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \frac{1}{8} n(n-1)(c+3) + (n-1) \text{trace}(h^T) + \frac{1}{2} \mathfrak{t}_M. \quad (6.10)$$

The equality cases of (6.10) is true if and only if p is a totally umbilical point.

Next, we have the following

Theorem 6.3. *Let M be an n -dimensional submanifold of a contact metric manifold \tilde{M} tangent to the structure vector field ξ . Then the following statements are true.*

(a) *If \tilde{M} is a (κ, μ) -space form $\tilde{M}(c)$ then the scalar inequality is given by*

$$\begin{aligned} \tau(p) \leq & \frac{n(n-1)}{2} \|H\|^2 + \frac{3}{8} (c-1) \|P\|^2 + \frac{1}{8} (n-1) (8\kappa + (n-2)(c+3)) \\ & + (\mu + n - 2) \text{trace}(h^T) + \frac{1}{2} \mathfrak{t}_M. \end{aligned} \quad (6.11)$$

(b) *If \tilde{M} is a non-Sasakian (κ, μ) -manifold then the scalar inequality is given by*

$$\begin{aligned} \tau(p) \leq & \frac{n(n-1)}{2} \|H\|^2 - \frac{3\mu}{4} \|P\|^2 + \frac{1}{4} (n-1) (4\kappa + (n-2)(2-\mu)) \\ & + (\mu + n - 2) \text{trace}(h^T) + \frac{1}{2} \mathfrak{t}_M. \end{aligned} \quad (6.12)$$

(c) *The equality cases of (6.11) (resp. (6.12)) are true if and only if p is a totally umbilical point.*

Proof. Using $2\tilde{\tau}(T_p M) = \sum_{i=1}^n \widetilde{\text{Ric}}_{(T_p M)}(e_i)$ from (5.5) and (6.5), it follows that

$$\begin{aligned} \tilde{\tau}(T_p M) = & \frac{1}{8} (n-1) (8\kappa + (n-2)(c+3)) \\ & + \frac{3}{8} (c-1) \|P\|^2 + (\mu + n - 2) \text{trace}(h^T) + \frac{1}{2} \mathfrak{t}_M. \end{aligned} \quad (6.13)$$

In view of (6.13) and (6.2) we get (6.11). Similarly, from (5.7) and (6.4) we get

$$\begin{aligned} \tilde{\tau}(T_p M) = & \frac{1}{4} (n-1) (4\kappa + (n-2)(2-\mu)) - \frac{3\mu}{4} \|P\|^2 \\ & + (\mu + n - 2) \text{trace}(h^T) + \frac{1}{2} \mathfrak{t}_M. \end{aligned} \quad (6.14)$$

Using (6.14) in (6.2) one obtains (6.12). \square

The inequality (6.12) is same as (3.8) of Theorem 3.3 in [47].

In view of Theorems 6.3 and 5.2 and the relations (5.8) we have the following

Theorem 6.4. *For an n -dimensional invariant submanifold M in a contact metric manifold \tilde{M} the following statements are true:*

(a) If \tilde{M} is a non-Sasakian (κ, μ) -space form $\tilde{M}(c)$ then

$$\tau = \frac{1}{8} (n-1) ((n+1)c + 8\kappa + 3(n-3)). \quad (6.15)$$

(b) If \tilde{M} is a non-Sasakian (κ, μ) -manifold then

$$\tau(p) = \frac{1}{4} (n-1) \{2n - (n+1)\mu + 4(\kappa-1)\} - \frac{1+\kappa-\mu}{2(1-\kappa)} \|h^T\|^2. \quad (6.16)$$

Remark 6.1. The part (b) of the above Theorem is an improvement of Theorem 7.1 of [47], where the scalar equality (6.16) is obtained as an inequality and a necessary and sufficient condition for equality case is given.

REFERENCES

- [1] V. I. Arnold: *Mathematical methods of classical mechanics*, Springer-Verlag, Berlin - Heidelberg, New York, 1989.
- [2] V. I. Arnold: *Contact geometry: the geometrical method of Gibb's thermodynamics*, Proceedings of the Gibbs Symposium, Yale University, (May 15-17, 1989), American Mathematical Society, 1990, 163-179.
- [3] K. Arslan, R. Ezentas, I. Mihai, C. Murathan and C. Ozgur: *Certain inequalities for submanifolds in (k, μ) -contact space forms*, Bull. Austral. Math. Soc., **64**(2001), No. 2, 201-212.
- [4] A. Bejancu: *Geometry of CR-submanifolds*, Mathematics and its Applications (East European Series) **23**, D. Reidel Publishing Co., Dordrecht, 1986.
- [5] D. E. Blair: *Two remarks on contact metric structures*, Tôhoku Math. J., **29**(1977), 319-324.
- [6] D. E. Blair: *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 203. Birkhauser Boston, Inc., Boston, MA, 2002.
- [7] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou: *Contact metric manifolds satisfying a nullity condition*, Israel J. Math., **91**(1995), No. 1-3, 189-214.
- [8] D. E. Blair, J. S. Kim and M. M. Tripathi: *On the concircular curvature tensor of a contact metric manifold*, J. Korean Math. Soc., **42**(2005), No. 5, 883-892.
- [9] E. Boeckx: *A class of locally φ -symmetric contact metric spaces*, Arch. Math. (Basel), **72**(1999), 466-472.
- [10] E. Boeckx: *A full classification of contact metric (κ, μ) -spaces*, Illinois J. Math., **44**(2000), No. 1, 212-219.
- [11] E. Boeckx: *Contact-homogeneous locally φ -symmetric manifolds*, Glasg. Math. J., **48**(2006), 93-109.
- [12] B.-Y. Chen: *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, Louvain, 1990, pp 123.
- [13] B.-Y. Chen: *Mean curvature and shape operator of isometric immersions in real space form*, Glasgow Math. J., **38**(1996), No. 1, 87-97.
- [14] B.-Y. Chen: *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*, Glasgow Math. J., **41**(1999), 33-41.

- [15] B.-Y. Chen: *On Ricci curvature of isotropic and Langrangian submanifolds in complex space forms*, Arch. Math. (Basel), **74**(2000), 154-160.
- [16] B.-Y. Chen: Riemannian submanifolds, in *Handbook of Differential Geometry* (F. Dillen and L. Verstraelen (Eds)), Vol. I, North Holland, Amsterdam, 2000, 187-418.
- [17] S. S. Chern: *Pseudo-groups continus infinis*, colloques Internationaux du C. N. R. S., Strassbourg, 1953, 119-136.
- [18] D. Cioroboiu: *Some inequalities for Ricci curvature of certain submanifolds in Sasakian space forms*, Acta Math. AcadPaedagog. Nyhazi., N. S., **19**(2003), No. 2, 233-243 (electronic).
- [19] H. Geiges: *A brief history of contact geometry and topology*, Expo. Math., **19**(2001), No. 1, 25-53.
- [20] J. W. Gray: *Some global properties of contact structures*, Ann. of Math., **69**(1959), 421-450.
- [21] S. P. Hong, K. Matsumoto and M. M. Tripathi: *Certain basic inequalities for submanifolds of locally conformal Kaehler space forms*, Sci. Univ. Tokyo J. Math., **41**(2005), No. 1, 75-94.
- [22] S. P. Hong and M. M. Tripathi: *On Ricci curvature of submanifolds*, Internat. J. Pure Appl. Math. Sci., **2**(2005), No. 2, 227-245.
- [23] S. P. Hong and M. M. Tripathi: *Ricci curvature of submanifolds of a Sasakian space form*, Iranian J. Math. Sci. Inform., **1**(2006), No. 2, 31-51.
- [24] N. Hurt: *Geometric quantization in action*, Reidel Publishing Company, Dordrecht, Boston, London, 1983.
- [25] J.-S. Kim, M. M. Tripathi and Jaedong Choi: *Ricci curvature of submanifolds in locally conformal almost cosymplectic manifolds*, Indian J. Pure Appl. Math., **35**(2004), No. 3, 259-271 (MR2046493 (2004m:53103)).
- [26] T. Koufogiorgos: *Contact metric manifolds*, Ann. Global Anal. Geom., **11**(1993), 25-34.
- [27] T. Koufogiorgos: *Contact Riemannian manifolds with constant φ -sectional curvature*, Tokyo J. Math., **20**(1997), 13-22.
- [28] X. Liu: *On Ricci curvature of C-totally real submanifolds in Sasakian space forms*, Acta Math. Acad. Paedagog. Nyhazi., N. S., **17**(2001), No. 3, 171-177 (electronic).
- [29] X. Liu: *On Ricci curvature of C-totally real submanifolds in Sasakian space forms*, Proc. Indian Acad. Sci. Math. Sci., **111**(2001), No. 4, 399-405.
- [30] X. Liu and X. Gao: *Note on a paper on Ricci curvature for odd-dimensional minimal integral submanifolds in a Sasakian space form $M^{2n+1}(c)$* , Sichuan Shifan Daxue Xuebao Ziran Kexue Ban, **24**(2001), No. 4, 397-398.
- [31] C. Lupu: *Chen-like inequalities for C-totally real submanifolds of Sasakian space forms*, Proceedings of the Centennial "G. Vrânceanu", Part II (Bucharest, 2000), An. Univ. București Mat. Inform., **50**(2001), No. 2, 83-86.
- [32] K. Matsumoto and I. Mihai: *Ricci tensor of C-totally real submanifolds in Sasakian space forms*, Nihonkai Math. J., **13**(2002), No. 2, 191-198.
- [33] K. Matsumoto, I. Mihai and A. Oiaga: *Ricci curvature of submanifolds in complex space forms*, Rev. Roumaine Math. Pures Appl., **46**(2001), No. 6, 775-782 (2002).

- [34] K. Matsumoto, I. Mihai and Y. Tazawa: *Ricci tensor of slant submanifolds in complex space forms*, Kodai Math. J., **26**(2003), No. 1, 85-94.
- [35] I. Mihai: *Ricci curvature of submanifolds in Sasakian space forms*, J. Aust. Math. Soc., **72**(2002), No. 2, 247-256.
- [36] B. C. Montano, L. Di Terlizzi and M. M. Tripathi: *Invariant submanifolds of contact (κ, μ) -manifolds*, Glasgow Math. J., **50**(2008), No. 3, 499-507.
- [37] J. F. Nash: *The imbedding problem for Riemannian manifolds*, Ann. of Math., **63**(1956), No. 2, 20-63.
- [38] K. Ogiue: *On almost contact manifolds admitting axiom of planes or axiom of free mobility*, Kōdai Math. Sem. Rep., **16**(1964), 223-232.
- [39] S. Sasaki: *On differentiable manifolds with certain structures which are closely related to almost contact structure. I*, Tôhoku Math. J., **12**(1960), No. 2, 459-476.
- [40] S. Sasaki: *On the differential geometry of tangent bundles of Riemannian manifolds II*, Tôhoku Math. J., **14**(1962), No. 2, 146-155.
- [41] S. Sasaki and Y. Hatakeyama: *On differentiable manifolds with certain structures which are closely related to almost contact structure. II*, Tôhoku Math. J., **13**(1961), No. 2, 281-294.
- [42] S. Sasaki and Y. Hatakeyama: *On differentiable manifolds with contact metric structures*, J. Math. Soc. Japan, **14**(1962), 249-271.
- [43] S. Tanno: *The topology of contact Riemannian manifolds*, Illinois J. Math., **12**(1968), 700-717.
- [44] S. Tanno: *Ricci curvatures of contact Riemannian manifolds*, Tôhoku Math. J., **40**(1988), 441-448.
- [45] Y. Tashiro: *On contact structure of tangent sphere bundles*, Tôhoku Math. J., **21**(1969), 117-143.
- [46] M. M. Tripathi: *Certain inequalities for submanifolds in locally conformal almost cosymplectic manifolds*, Trends in Math., Information Center for Mathematical Sciences, **8**(2002), No. 2, 87-96.
- [47] M. M. Tripathi: *Certain basic inequalities for submanifolds in (κ, μ) -spaces*, Contemporary Math. (American Mathematical Society), **337**(2003), 187-202.
- [48] M. M. Tripathi: *Certain inequalities for submanifolds in cosymplectic space forms*, International Conference on Mathematics and its Applications (ICMA 2004), Kuwait University, Kuwait, Extended Abstracts, 284-294.
- [49] M. M. Tripathi and J.-S. Kim: *C-totally real submanifolds in (κ, μ) -contact space forms*, Bull. Austral. Math. Soc., **67**(2003), No. 1, 51-65.
- [50] M. M. Tripathi, J.-S. Kim and S.-B. Kim: *Mean curvature and shape operator of slant immersions in a Sasakian space form*, Balkan J. Geom. Appl., **7**(2002), No. 1, 101-111.
- [51] D. Van Lindt, P. Verheyen and L. Verstraelen: *Minimal submanifolds in Sasakian space forms*, J. Geom., **27**(1986), No. 2, 180-187.
- [52] H. Wu: *Manifolds of partially positive curvature*, Indiana Univ. Math. J., **36**(1987), No. 3, 525-548.

- [53] S. Yamaguchi, M. Kon and T. Ikawa, *C-totally real submanifolds*, J. Differential Geometry, **11**(1976), No. 1, 59-64.
- [54] K. Yano and M. Kon: *Anti-invariant submanifolds*, Lecture Notes in Pure and Applied Mathematics, 21. Marcel Dekker, Inc., New York, Basel, 1976.
- [55] K. Yano and M. Kon: *Structures on manifolds*, Series in Pure Mathematics, 3. World Scientific Publishing Co., Singapore, 1984.

Banaras Hindu University
Faculty of Science
Department of Mathematics
Varanasi 221005, India
E-mail address: mmtripathi66@yahoo.com