J. Adv. Math. Studies
Vol. **1**(2008), No. 1-2, 97-110
`http://journal.fairpartners.ro`

HIGHER ORDER GEOMETRY ON ALMOST LIE STRUCTURES

PAUL POPESCU AND MARCELA POPESCU

ABSTRACT. Lifting of almost Lie structures to higher order using connections are constructed. In the case of algebroids and Lie algebroids the lifts are canonical.

1. INTRODUCTION

The present paper is a natural continuation of [18], where the basic geometric structures of higher order anchored vector bundles (vertical vector bundles, connections, semi-sprays) are defined. The next step is considered in this paper, where the almost Lie structures of higher order are constructed. The relation between the almost Lie structures of higher order and the vector bundles constructed in [18] are related by some isomorphisms which depend on the almost Lie structures, in the case of order one and two; the general case is still a mystery for us.

An *anchored vector bundle (AVB)* (or a relative tangent space, see [14, 17]) is a couple (θ, D), where $\theta = (E, p, M)$ is a vector bundle and $D\colon \theta \to \tau M$ is a vector bundle morphism called an *anchor* (an arrow, or a tangent map). The vector bundle $\tau M = (TM, p_0, M)$ is the tangent bundle of M. For example, if θ is a vector subbundle of τM and $i\colon \theta \to \tau M$ is the inclusion morphism, then (θ, i) is an AVB.

A *bracket* (or a Lie map) on an AVB (θ, D) is a map $[\cdot,\cdot]\colon \Gamma(\theta)\times\Gamma(\theta)\to\Gamma(\theta)$ which enjoys the properties that it is bilinear over $\mathbb{R}$, is skew symmetric and

$$[X, fY] = (DX)(f)Y + f[X, Y], \quad (\forall)X, Y \in \Gamma(\theta) \text{ and} f \in \mathcal{F}(M).$$

An *almost Lie structure* (ALS) is a triple $(\theta, D, [\cdot,\cdot]_\theta)$. Linear and non-linear connections associated with AVB's and ALS's are studied for the first time in [14, 15], where they are called R-connections. The (non)linear R-connections are known as Wong connections [20] and they were recently considered for example in [5] and [6].

An *algebroid* is an ALS $(\theta, D, [\cdot,\cdot]_\theta)$ which enjoys the property that $[DX, DY] = D[X, Y]_\theta$, $(\forall)X, Y \in \Gamma(\theta)$, where the first bracket is the Lie bracket on $\mathcal{X}(M)$. A *Lie algebroid* is an algebroid $(\theta, D, [\cdot,\cdot]_\theta)$ which has a null Jacobiator, i.e.

$$\mathcal{J}(X, Y, Z) \equiv \sum_{\text{cicl.}} \left[[X, Y]_\theta, Z\right]_\theta = 0, \quad (\forall)X, Y, Z \in \Gamma(\theta).$$

Received: December 05, 2008.

2000 *Mathematics Subject Classification*: 53C07, 44A15, 53B15, 22A30, 70S05, 53C05, 53C60.

Key words and phrases: Anchored vector bundle, almost Lie structure, affine bundle, r-prolongation, R-connection.

The linear R-connections defined using Lie algebroids are used in [4] (where they are called A-connections).

A theory of higher order spaces and geometric structures is considered in [7, 9]. We use the point of view of these papers concerning higher order extensions, and also the local calculus, which is used in the mainly part of the basic constructions.

We consider below some cases where the constructions in our paper applies, the same as in [18].

1. Let M be a smooth manifold. Any subbundle $\theta \subset \tau M$ defines an anchored vector bundle.

2. A Lie algebroid.

3. Let M be a smooth manifold. Let M be a smooth manifold and

$$\{\cdot,\cdot\}\colon \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$$

be a Poisson bracket. It is well-known that the association $df \to X_f$ (the hamiltonian vector field which correspond to a real function $f \in \mathcal{F}(M)$ is given by $X_f(g) = \{f, g\}$) extends to an anchor $D\colon \tau^*M \to \tau M$. The bracket of functions defines also by extension a bracket of real forms. A Lie algebroid is defined in this way on τ^*M.

4. Let M be a smooth manifold. A two-vector field $P \in \Lambda_2(M)$ define an anchor $D : \tau^*M \to \tau M$. In particular, if ω comes from the inverse of a symplectic form on M, then a (non-singular) Poisson bracket follows (thus **3** applies).

5. Let M be a smooth manifold. The canonical symplectic form on T^*M defines an anchor $D\colon \tau^*T^*M \to \tau T^*M$ (thus **4** applies).

The first section contains the basic constructions of the prolongations of order $r \geq 2$ of an AVB and the other vector bundles associated with these prolongations, as studied in the first section of [18]. Anchored vector bundles, connections and almost Lie structures of higher order, as well as isomorphisms of the vector bundles $\xi^{(r)}$ and $\eta^{(r)}$ for $r = 1, 2$, depending on the almost Lie structures, are considered in the second section. The Appendix explains the simple construction of an isomorphism of the middle therm of two exact sequences of vector bundles, used in the paper.

2. THE R-PROLONGATIONS OF AN ANCHORED VECTOR BUNDLE

Let (θ, D) be an anchored vector bundle. We define $\theta^{(0)} = M$, i.e. the trivial vector bundle over M, and $\theta^{(1)} = \theta = (E, \pi, M) = (E^{(1)}, \pi^{(1)}, E^{(0)})$. Assume that $\{g_\alpha^{\alpha'}(x^i)\}$ is the cocycle (structural functions) of E. Thus, on the intersection of two domains of two adapted charts, the local coordinates on $E^{(1)}$ change according to the rules

$$x^{i'} = x^{i'}(x^j), \quad y^{(1)\alpha'} = g_\alpha^{\alpha'} y^{(1)\alpha}.$$

Consider now the local vector field on $E^{(1)}$ defined by $\Gamma^{(1)} = y^{(1)\beta} D_\beta^i \dfrac{\partial}{\partial x^i}$. The association to an open domain of the adapted coordinates with the local vector field is called in [19] a vector pseudofield. We define the affine bundle $\theta^{(2)} = (E^{(2)}, \pi^{(2)}, E^{(1)})$ using, for the coordinates on the fibers, the change rules

$$2y^{(2)\alpha'} = 2g_\alpha^{\alpha'} y^{(2)\alpha} + \Gamma^{(1)}(y^{(1)\alpha'}).$$

Assuming, for $r \geq 2$, that the affine bundles $\theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(r)}=(E^{(r-1)}, \pi^{(r-1)}, E^{(r-2)})$ and the vector pseudofields $\Gamma^{(0)}, \Gamma^{(1)}, \ldots, \Gamma^{(r-1)}$ has been defined, we define:

(1) the affine bundle $\theta^{(r)} = (E^{(r)}, \pi^{(r)}, E^{(r-1)})$, according to the formula:

$$ry^{(r)\alpha'} = \Gamma^{(r-1)}(y^{(r-1)\alpha'}) + ry^{(r)\beta} g_{\alpha}^{\alpha'}$$

for the change rule of the coordinates on the fibers of $E^{(r)}$and

(2) the vector pseudofield $\Gamma^{(r)}$ (on $E^{(r)}$) according to the formula

$$\Gamma^{(r)} = \Gamma^{(r-1)} + y^{(r)\beta} \frac{\partial}{\partial y^{(r-1)\beta}}.$$

It is easy to see that $\dfrac{\partial y^{(r)\alpha'}}{\partial y^{(r)\alpha}} = g_{\alpha}^{\alpha'}$, $(\forall) r \geq 1$.

It follows that the change rule of the coordinates on the manifold $E^{(r)}$ is:

$$\begin{aligned} y^{(1)\alpha'} &= g_{\beta}^{\alpha} y^{(1)\beta}, \\ 2y^{(2)\alpha'} &= y^{(1)\beta} D_{\beta}^{i} \frac{\partial y^{(1)\alpha'}}{\partial x^i} + 2 \frac{\partial y^{(1)\alpha'}}{\partial y^{(1)\beta}} y^{(2)\beta}, \\ &\vdots \\ ry^{(r)\alpha'} &= y^{(1)\beta} D_{\beta}^{i} \frac{\partial y^{(r-1)\alpha'}}{\partial x^i} + 2y^{(2)\beta} \frac{\partial y^{(r-1)\alpha'}}{\partial y^{(1)\beta}} + \cdots + \\ &\qquad + (r-1) y^{(r-1)\beta} \frac{\partial y^{(r-1)\alpha'}}{\partial y^{(r-2)\beta}} + ry^{(r)\beta} \frac{\partial y^{(r)\alpha'}}{\partial y^{(r)\beta}}. \end{aligned}$$

Proposition 2.1. *The bundles $\theta^{(r)} = (E^{(r)}, \pi^{(r)}, E^{(r-1)})$, $r \geq 1$, are affine bundles.*

Proof. See [18, Proposition 1.1]. □

The local adapted coordinates on $E^{(r)}$, $r \geq 0$, used above, are called *adapted.*

An important particular case is obtained when $\theta = \tau M$ is the tangent bundle of M (i.e. $E = TM$) and the anchor $D = 1_{TM}$ is the identity. In this case $(\tau M)^{(n)} = T^{(n)} M$ is the total space of the acceleration bundle of order k, studied for example in [8, 9, 7].

Notice that for every $r \geq 1$, using local coordinates which are adapted, the natural local basis in $TE^{(r)}$ change according to the rules:

$$\begin{aligned} \frac{\partial}{\partial x^i} &= \frac{\partial x^{j'}}{\partial x^i} \frac{\partial}{\partial x^{j'}} + \frac{\partial y^{(1)\beta'}}{\partial x^i} \frac{\partial}{\partial y^{(1)\beta'}} + \cdots + \frac{\partial y^{(r)\beta'}}{\partial x^i} \frac{\partial}{\partial y^{(r)\beta'}}, \\ \frac{\partial}{\partial y^{(1)\alpha}} &= \frac{\partial y^{(1)\beta'}}{\partial y^{(1)\alpha}} \frac{\partial}{\partial y^{(1)\beta'}} + \cdots + \frac{\partial y^{(r)\beta'}}{\partial y^{(1)\alpha}} \frac{\partial}{\partial y^{(r)\beta'}}, \\ &\vdots \\ \frac{\partial}{\partial y^{(r)\alpha}} &= \frac{\partial y^{(r)\beta'}}{\partial y^{(r)\alpha}} \frac{\partial}{\partial y^{(r)\beta'}}. \end{aligned} \tag{2.1}$$

On the intersection of two domains of coordinates one has:

$$\bar{\Gamma}^{(r)} = \Gamma^{(r)} - \Gamma^{(r)}(y^{(r)\alpha'})\frac{\partial}{\partial y^{(r)\alpha'}}. \tag{2.2}$$

Indeed,

$$\begin{aligned}
\Gamma^{(r)} &= y^{(1)\beta}D^i_\beta\frac{\partial}{\partial x^i} + 2y^{(2)\beta}\frac{\partial}{\partial y^{(1)\beta}} + \cdots + ry^{(r)\beta}\frac{\partial}{\partial y^{(r-1)\beta}} \\
&= y^{(1)\beta}D^i_\beta\left(\frac{\partial x^{j'}}{\partial x^i}\frac{\partial}{\partial x^{j'}} + \frac{\partial y^{(1)\beta'}}{\partial x^i}\frac{\partial}{\partial y^{(1)\beta'}} + \cdots + \frac{\partial y^{(r)\beta'}}{\partial x^i}\frac{\partial}{\partial y^{(r)\beta'}}\right) \\
&\quad +2y^{(2)\beta}\left(\frac{\partial y^{(1)\beta'}}{\partial y^{(1)\alpha}}\frac{\partial}{\partial y^{(1)\beta'}} + \cdots + \frac{\partial y^{(r)\beta'}}{\partial y^{(1)\alpha}}\frac{\partial}{\partial y^{(r)\beta'}}\right) + \cdots + \\
&\quad +(r-1)y^{(r-1)\beta}\left(\frac{\partial y^{(r-1)\beta'}}{\partial y^{(r-1)\alpha}}\frac{\partial}{\partial y^{r-(1)\beta'}} + \cdots + \frac{\partial y^{(r)\beta'}}{\partial y^{(r-1)\alpha}}\frac{\partial}{\partial y^{(r)\beta'}}\right) \\
&\quad +ry^{(r)\alpha}g^{\beta'}_\alpha\frac{\partial}{\partial y^{(r)\beta'}} = y^{(1)\beta'}D^{i'}_{\beta'}\frac{\partial}{\partial x^{i'}} + 2y^{(2)\beta'}\frac{\partial}{\partial y^{(1)\beta'}} + \cdots + ry^{(r)\beta'}\frac{\partial}{\partial y^{(r-1)\beta'}} \\
&\quad +\Gamma^{(r)}(y^{(r)\beta'})\frac{\partial}{\partial y^{(r)\beta'}} = \bar{\Gamma}^{(r)} + \Gamma^{(r)}(y^{(r)\beta'})\frac{\partial}{\partial y^{(r)\beta'}}.
\end{aligned}$$

Lemma 2.1. *The equality*

$$\frac{\partial y^{(r+1)\alpha'}}{\partial y^{(u)\beta}} = \frac{\partial y^{(r)\alpha'}}{\partial y^{(u-1)\beta}}, \tag{2.3}$$

holds for $2 \leq u \leq r+1$.

Proof. See [18, Lemma 1.1]. □

3. THE HIGHER ORDER ANCHORED VECTOR BUNDLES AND ALMOST LIE STRUCTURES

We call the vector subbundle $\ker \pi^{(r)}_* \subset \tau E^{(r)}$ the *partial r-vertical bundle* of the anchored vector bundle (θ, D). We call it *partial* in order to distinct from a total r-vertical bundle, defined as follows.

Let us consider the sequence of applications

$$E^{(r)} \overset{\pi^{(r)}}{\to} E^{(r-1)} \overset{\pi^{(r-1)}}{\to} E^{(r-2)} \to \cdots \to E^{(1)} \overset{\pi^{(1)}}{\to} E^{(0)} = M.$$

Then $\Pi^{(r)} = \pi^{(1)} \circ \pi^{(2)} \circ \cdots \circ \pi^{(r)}$ define a fibered manifold $(E^{(r)}, \Pi^{(r)}, M)$ over the base M, for every $r \geq 1$. We say that the vector subbundle $V\theta^{(r)} = \ker \Pi^{(r)}_* \subset \tau E^{(r)}$ is the *total r-vertical bundle*.

Proposition 3.1 ([18], Proposition 1.1)**.** *There is a vector bundle*

$$\xi^{(r)} = (F^{(r)}, p^{(r)}_0, E^{(r)}),\ \text{over } E^{(r)},$$

such that the total r-vertical bundle $V\theta^{(r+1)}$*, over* $E^{(r+1)}$*, is canonically isomorphic with the vector bundle* $\left(\pi^{(r+1)}\right)^{*}\xi^{(r)}$.

Proof. On the same intersection domains

$$\left(\Pi^{(r+1)}\right)^{-1}(U)\cap\left(\Pi^{(r+1)}\right)^{-1}(\bar{U})=\left(\Pi^{(r+1)}\right)^{-1}(U\cap\bar{U})\subset E^{(r+1)},$$

of local adapted charts on $E^{(r+1)}$, the structural functions of the vertical bundle $\ker\Pi_{*}^{(r+1)}$ are the local matrices:

$$\overline{\mathcal{M}}_{\bar{U}\cap U}=\begin{pmatrix}\dfrac{\partial y^{(1)\alpha_0'}}{\partial y^{(1)\beta_0}} & 0 & \cdots & 0\\ \dfrac{\partial y^{(2)\alpha_1'}}{\partial y^{(1)\beta_0}} & \dfrac{\partial y^{(2)\alpha_1'}}{\partial y^{(2)\beta_1}} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ \dfrac{\partial y^{(r+1)\alpha_r'}}{\partial y^{(1)\beta_0}} & \dfrac{\partial y^{(r+1)\alpha_r'}}{\partial y^{(2)\beta_1}} & \cdots & \dfrac{\partial y^{(r+1)\alpha_r'}}{\partial y^{(r+1)\beta_r}}\end{pmatrix}$$

$$=\begin{pmatrix}\dfrac{\partial y^{(1)\alpha_0'}}{\partial y^{(1)\beta_0}} & 0 & \cdots & 0\\ \dfrac{\partial y^{(2)\alpha_1'}}{\partial y^{(1)\beta_0}} & \dfrac{\partial y^{(1)\alpha_1'}}{\partial y^{(1)\beta_1}} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ \dfrac{\partial y^{(r+1)\alpha_r'}}{\partial y^{(1)\beta_0}} & \dfrac{\partial y^{(r)\alpha_r'}}{\partial y^{(1)\beta_1}} & \cdots & \dfrac{\partial y^{(r)\alpha_r'}}{\partial y^{(r)\beta_r}}\end{pmatrix} \tag{3.1}$$

It is easy to see that the component local functions of these matrices do not depend on the variables $\{y^{(r+1)\alpha},\ \alpha=\overline{1,m}\}$, thus the matrices $\overline{\mathcal{M}}_{\bar{U}\cap U}$ are induced by the local matrices $\mathcal{M}_{\bar{U}\cap U}$, which have the same form, but defined on the intersection domains $\left(\Pi^{(r)}\right)^{-1}(U)\cap\left(\Pi^{(r)}\right)^{-1}(\bar{U})=\left(\Pi^{(r)}\right)^{-1}(U\cap\bar{U})\subset E^{(r)}$. Thus there is a vector bundle $\xi^{(r)}=(F^{(r)},p_0^{(r)},E^{(r)})$, over $E^{(r)}$, as required. □

3.1. **Anchored vector bundles of higher order.** Let us consider an other anchored vector bundle (μ,ρ), where $\mu=(Q,q,M)$ is a vector bundle and $\rho\colon\mu\to\tau M$ is the anchor. Let consider the fibered product $TE^{(r)}\times_{TM}Q$, the canonical projection $\Lambda^{(r)}\colon TE^{(r)}\times_{TM}Q\to TE^{(r)}$ and $\lambda^{(r)}\colon TE^{(r)}\times_{TM}Q\to E^{(r)}$ obtained as $\lambda^{(r)}=u^{(r)}\circ\Lambda^{(r)}$, where $u^{(r)}\colon TE^{(r)}\to E^{(r)}$ is the canonical projection.

Proposition 3.2. *There is a vector bundle* $\eta_{\mu}^{(r)}=(TE^{(r)}\times_{TM}Q,\lambda^{(r)},E^{(r)})$ *and* $(\eta_{\mu}^{(r)},\Lambda^{(r)})$ *is an anchored vector bundle.*

If $\mu=\theta$ and $\rho=D$ (i.e. (μ,ρ) is the initial anchor bundle (θ,D)), then we denote $\eta_{\mu}^{(r)}=\eta^{(r)}$. It follows the anchored vector bundle $(\eta^{(r)},\Lambda^{(r)})$.

An other important case arise for an almost Lie structure. Let us suppose that $(\theta, D, [\cdot,\cdot]_\theta)$ is an almost Lie structure on θ. The local components of the bracket $[\cdot,\cdot]_\theta$ are the local functions $\{B^\alpha_{\beta\gamma}\}$ on E (the total space of θ), defined by the relations $[s_\beta, s_\gamma]_\theta = B^\alpha_{\beta\gamma} s_\alpha$. The formula

$$\Gamma(\theta\otimes\theta) \ni X\otimes Y \to \mathcal{D}(X,Y) = [D(X), D(Y)] - D([X,Y]_\theta) \in \mathcal{X}(M)$$

define an anchor on the vector bundle $\theta\otimes\theta$ (since $\mathcal{D}$ is skew symmetric, it can be considered, as well, a bracket on the exterior product vector bundle $\theta\wedge\theta$). If the almost Lie structure is an algebroid or a Lie algebroid, then $\mathcal{D}=0$. It follows the anchored vector bundle $(\theta\wedge\theta, \mathcal{D})$.

Using local coordinates, the local structural functions of the vector bundle $\eta^{(r)}_\mu$ on a domain $\left(\Pi^{(r)}\right)^{-1}(U) \cap \left(\Pi^{(r)}\right)^{-1}(\bar U) = \left(\Pi^{(r)}\right)^{-1}(U\cap\bar U) \subset E^{(r)}$ are

$$C^{(r)}_{\bar U\cap U} = \begin{pmatrix} h^a_b & 0 & \cdots & 0 \\ \rho^i_b \dfrac{\partial y^{(1)\alpha'_1}}{\partial x^i} & \dfrac{\partial y^{(1)\alpha'_1}}{\partial y^{(1)\beta_1}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \rho^i_b \dfrac{\partial y^{(r)\alpha'_1}}{\partial x^i} & \dfrac{\partial y^{(r)\alpha'_r}}{\partial y^{(1)\beta_1}} & \cdots & \dfrac{\partial y^{(r)\alpha'_r}}{\partial y^{(r)\beta_r}} \end{pmatrix}, \tag{3.2}$$

where $(h^a_b)_{a,b=\overline{1,m_1}}$ are the structural functions on $U\subset M$.

For $\eta^{(r)}$ the structural functions have the form:

$$A_{\bar U\cap U} = \begin{pmatrix} g^{\alpha'_0}_{\beta_0} & 0 & \cdots & 0 \\ D^i_{\beta_0} \dfrac{\partial y^{(1)\alpha'_1}}{\partial x^i} & \dfrac{\partial y^{(1)\alpha'_1}}{\partial y^{(1)\beta_1}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ D^i_{\beta_0} \dfrac{\partial y^{(r)\alpha'_1}}{\partial x^i} & \dfrac{\partial y^{(r)\alpha'_r}}{\partial y^{(1)\beta_1}} & \cdots & \dfrac{\partial y^{(r)\alpha'_r}}{\partial y^{(r)\beta_r}} \end{pmatrix}. \tag{3.3}$$

For every $r\geq 1$ there is a natural $\pi^{(r)}$-epimorphism $p^{(r)}\colon \xi^{(r)} \to \xi^{(r-1)}$ of vector bundles. In an analogous way one can define an $\pi^{(r)}$-epimorphism $q^{(r)}\colon \eta^{(r)} \to \eta^{(r-1)}$. The vector subbundles $\ker p^{(r)} \subset \xi^{(r)}$ and $\ker q^{(r)} \subset \eta^{(r)}$ are both isomorphic with the vector subbundle $\ker \pi^{(r)}_* \subset \tau E^{(r)}$, called here the *partial r-vertical bundle* of the anchored vector bundle (θ, D). We call it *partial* in order to distinct from the total r-vertical bundle, defined previously. We can also consider, for $u=\overline{1,r}$, $\Pi^{(r)}_u = \pi^{(u)}\circ\pi^{(u+1)}\circ\cdots\circ\pi^{(r)}$ and the vector subbundles

$$\ker\left(\Pi^{(r)}_r\right)_* \subset \ker\left(\Pi^{(r)}_{r-1}\right)_* \subset\cdots\subset \ker\left(\Pi^{(r)}_1\right)_* = \ker\Pi^{(r)}_* \subset \tau E^{(r)}$$

(they are involutive subbundles of $\tau E^{(r)}$). Analogously to the partial r-vertical bundle, these vector bundles can be also obtained as a subbundles of $\xi^{(r)}$ and $\eta^{(r)}$, as follows. The vector subbundle $\ker P^{(r)}_u \subset \xi^{(r)}$, for $1\leq u\leq r$, defined by the vector bundle epimorphism $P_{(u)}^{(r)} = p^{(u)} \circ \cdots \circ p^{(r)} \colon E^{(r)} \to E^{(1)} = E$, is obviously isomorphic with the total r-vertical bundle $\ker \left(\Pi_u^{(r)}\right)_*$; we denote it as $V\theta_{(u)}^{(r)}$. Analogously one can consider the vector subbundle $V\eta_{(u)}^{(r)}$, isomorphic with the vector subbundle $V\theta_{(u)}^{(r)}$. for every $1 \le u \le r$. Notice that:

(1) the canonically isomorphic vector bundles $V\theta_{(1)}^{(r)} \overset{\text{not.}}{=} V\theta^{(r)}$ and $V\eta_{(1)}^{(r)} \overset{\text{not.}}{=} V\eta^{(r)}$ are isomorphic with the total r-vertical bundle and
(2) for any anchored vector bundle (μ, ρ), one can consider the total r-partial bundle $V\eta_{\mu(u)}^{(r)}$, isomorphic with the vector bundle $V\eta_{(u)}^{(r)}$; particularly the vector bundle $V\eta_{\mu(1)}^{(r)}$ is isomorphic with the total r-vertical bundle.

In the case when $\theta = \tau M$ is the tangent bundle of M, the vector bundles $\xi^{(r)}$ and $\eta^{(r)}$, are equal. In general they are different, but isomorphic (see 4.1 in Appendix). Nevertheless, there is not a canonical isomorphism of these vector bundles. In the next section we show that for $r = 1$ and $r = 2$, an adapted almost Lie structure on $\xi^{(r)}$ define an isomorphism of the vector bundles $\xi^{(r)}$ and $\eta^{(r)}$. It seems that this fact is also true for $r \ge 3$, but we have not yet a proof.

3.2. **Connections on anchored vector bundles.** In the sequel we use the previous notations.

A *connection* on $\eta_\mu^{(r)}$ is equivalently given by

(1) a left splitting of the inclusion $i \colon V\theta^{(r)} \cong V\eta_{\mu(1)}^{(r)} \to \eta_\mu^{(r)}$ of the total r-vertical subbundle (i.e. a vector bundle map $C \colon \eta_\mu^{(r)} \to V\theta^{(r)}$, called the *$r$-connection map*) such that $C \circ i = 1_{V\theta^{(r)}}$) or
(2) a vector subbundle $H\eta_\mu^{(r)}$ of the vector bundle $\eta_\mu^{(r)}$ (called the *r-horizontal bundle*), such that $\eta_\mu^{(r)} = V\theta^{(r)} \oplus H\eta_\mu^{(r)}$ (Whitney sum).

The link between the connection map C and the horizontal vector bundle $H\theta^{(r)}$ is $H\theta^{(r)} = \ker C$.

In local coordinates, consider a base $\{s_{(0)a}, s_{(1)\alpha}, \ldots, s_{(r)\alpha}\}_{a=\overline{1,m_1}, \alpha=\overline{1,m}}$ of local sections on $\eta_\mu^{(r)}$ given by some adapted coordinates, corresponding to the structural functions (3.2).

The connection map C has the local form:

$$C(s_{(0)a}) = \underset{1}{N}{}_a^\beta s_{(1)\beta} + \cdots + \underset{r}{N}{}_a^\beta s_{(r)\beta}, C(s_{(1)\alpha}) = s_{(1)\alpha}, \ldots, C(s_{(r)\alpha}) = s_{(r)\alpha}.$$

The local functions $\underset{1}{N}{}_a^\beta, \ldots \underset{r}{N}{}_a^\beta$ are called the *local coefficients* of the connection.

It follows that the local sections $\{t_{(0)a}\}_{a=\overline{1,m_1}}$ defined by

$$t_{(0)a} = s_{(0)a} - \underset{1}{N}{}_a^\beta s_{(1)\beta} - \cdots - \underset{r}{N}{}_a^\beta s_{(r)\beta}$$

are a local base of the sections on $H\eta_\mu^{(r)}$. On the intersection of two local domains on $E^{(r)}$, these local sections change according to the rule $t_{(0)a} = h_a^b\,\bar{t}_{(0)b}$. Notice that the r-horizontal vector bundle $H\theta^{(r)}$ is isomorphic with each of the following vector bundles:

(1) the quotient vector bundle $\eta_\mu^{(r)}/V\theta^{(r)}$ and
(2) the induced vector bundle $(\Pi^{(r)})^*\mu$, where $\Pi^{(r)}\colon E^{(r)} \to M$ is the canonical projection.

Two coordinate systems change on $TE^{(r)} \times_{TM} Q$ (the total space of $\eta_\mu^{(r)} = (TE^{(r)} \times_{TM} Q, \lambda^{(r)}, E^{(r)})$) according to the rules

$$\begin{aligned} s_{(0)a} &= h_a^{b'}\bar{s}_{(0)b'} + \rho_a^i\frac{\partial \bar{y}^{(1)\beta'}}{\partial x^i}\bar{s}_{(1)\beta'} + \cdots + \rho_a^i\frac{\partial \bar{y}^{(r)\beta'}}{\partial x^i}\bar{s}_{(r)\beta'},\\ s_{(1)\alpha} &= \frac{\partial \bar{y}^{(1)\beta'}}{\partial y^{(1)\alpha}}\bar{s}_{(1)\beta'} + \cdots + \frac{\partial \bar{y}^{(r)\beta'}}{\partial y^{(1)\alpha}}\bar{s}_{(r)\beta'},\\ &\vdots\\ s_{(r)\alpha} &= \frac{\partial \bar{y}^{(r)\beta'}}{\partial y^{(r)\alpha}}\bar{s}_{(r)\beta'}. \end{aligned} \tag{3.4}$$

Hence, the local components $\underset{1}{N}{}_a^\beta, \ldots, \underset{r}{N}{}_a^\beta$ of a connection change according to the rules:

$$\begin{aligned} h_a^{a'}\underset{1}{N}{}_{a'}^{\alpha'} &= \frac{\partial \bar{y}^{(1)\alpha'}}{\partial y^{(1)\alpha}}\underset{1}{N}{}_a^\alpha - \rho_a^i\frac{\partial \bar{y}^{(1)\alpha'}}{\partial x^i},\\ h_a^{a'}\underset{2}{N}{}_{a'}^{\alpha'} &= \frac{\partial \bar{y}^{(2)\alpha'}}{\partial y^{(2)\alpha}}\underset{2}{N}{}_a^\alpha + \frac{\partial \bar{y}^{(2)\alpha'}}{\partial y^{(1)\alpha}}\underset{1}{N}{}_a^\alpha - \rho_a^i\frac{\partial \bar{y}^{(2)\alpha'}}{\partial x^i},\\ &\vdots\\ h_a^{a'}\underset{r}{N}{}_{a'}^{\alpha'} &= \frac{\partial \bar{y}^{(r)\alpha'}}{\partial y^{(r)\alpha}}\underset{r}{N}{}_a^\alpha + \cdots + \frac{\partial \bar{y}^{(r)\alpha'}}{\partial y^{(1)\alpha}}\underset{1}{N}{}_a^\alpha - \rho_a^i\frac{\partial \bar{y}^{(r)\alpha'}}{\partial x^i}. \end{aligned}$$

The above formulas show that it is possible that a connection be constructed inductively. In this case, for a given $1 \le k \le r$, the local functions $\{\underset{k}{N}{}_a^\alpha\}$ depend only on the coordinates $\{x^i, y^{(1)\alpha}, \ldots, y^{(k)\alpha}\}_{i=\overline{1,n},\alpha=\overline{1,m}}$.

3.3. **Almost Lie structures of higher order.** We briefly recall the constructions performed in [14, 15, 16], which are extended in this subsection.

Let (D,θ) be an anchored vector bundle, $\theta = (E,p,M)$ and $D\colon \theta \to \tau M = (TM, p_0, M)$. An R-connection defined by (D,θ) on an fibered manifold $\xi = (F,\pi,M)$ was considered in [14]. In particular ξ may be a vector bundle as in [16]. Consider the vector bundle $E\xi = (EF, p_1, F)$, over the base F, which has as total space $EF = E \times_{TM} TF$, i.e. the fibered product obtained using $D\colon E \to TM$

and $\pi_*\colon TF \to TM$. The natural projection $\Delta\colon E \times_{TM} TF \to TF$ becomes an anchor. Using local coordinates (x^i) on $U \subset M$, (x^i, X^α) on $p^{-1}(U) \subset E$, (x^i, y^a) on $\pi^{-1}(U) \subset F$ and $(x^i, y^a, X^\alpha, Y^b)$ on $(\pi \circ p_1)^{-1}(U) \subset EF$, the local form of Δ is given by the local matrices

$$\Delta_U = \begin{pmatrix} D^i_\alpha & 0 \\ 0 & \delta^a_b \end{pmatrix}.$$

The natural projection $\Pi\colon EF \to F$ defines an epimorphism of vector bundles $\Pi\colon E\xi \to \theta$. The vertical bundle $V\xi = \ker \pi_*$ and the vector bundle $\ker \Pi = VE\xi$ are isomorphic. A left splitting $C\colon E\xi \to VE\xi$ of the inclusion $i\colon VE\xi \to E\xi$ is an R-connection on ξ.

In [15] was defined an *adapted* almost Lie structure, corresponding to the anchor Δ, having the form as follows. Let us denote as $\{s_\alpha\}_{\alpha=\overline{1,m}}$ the local base of sections on θ and $\{\bar{s}_\alpha, S_\beta\}_{\alpha,\beta=\overline{1,m}}$ the local base of sections on $E\xi$. Let us suppose that $(\theta, D, [\cdot,\cdot]_\theta)$ is an almost Lie structure on θ. The local components of the bracket $[\cdot,\cdot]_\theta$ are the local functions $\{B^\alpha_{\beta\gamma}\}$ on E (the total space of θ), defined by the relations $[s_\beta, s_\gamma]_\theta = B^\alpha_{\beta\gamma} s_\alpha$. The formula

$$\Gamma(\theta \otimes \theta) \ni X \otimes Y \to \mathcal{D}(X,Y) = [D(X), D(Y)] - D([X,Y]_\theta) \in \mathcal{X}(M)$$

define an anchor on the vector bundle $\theta \otimes \theta$ (since $\mathcal{D}$ is skew symmetric, it can be considered, as well, a bracket on the exterior product vector bundle $\theta \wedge \theta$). If the almost Lie structure is an algebroid or a Lie algebroid, then $\mathcal{D} = 0$. According to [15, 16], an *adapted almost Lie structure* on $E\xi$ is defined by an R-connection on θ, using the anchored vector bundle $(\theta \otimes \theta, \mathcal{D})$. Using local adapted coordinates, if $C^\alpha_{\beta\gamma}(x^i, y^{(1)\alpha})$ are the local components of the R-connection, then

$$[\bar{s}_\beta, \bar{s}_\gamma]_{E\xi} = B^\alpha_{\beta\gamma} \bar{s}_\alpha + C^\alpha_{\beta\gamma} S_\alpha, \quad [\bar{s}_\beta, S_\gamma]_{E\xi} = [S_\beta, S_\gamma]_{E\xi} = 0.$$

If θ is an algebroid or a Lie algebroid (i.e. $\mathcal{D} = 0$), then one take $C^\alpha_{\beta\gamma} = 0$; in fact it is the only situation when these coefficients vanish.

We consider now the general case $r \geq 1$. We are going to construct brackets $[\cdot,\cdot]_{\eta^{(r)}}$, using the anchor $\Delta^{(r)}$ on $\eta^{(r)}$.

The case $r = 1$ is that of the almost Lie structure $(\theta = \eta^{(1)}, D = \Delta^{(1)}, [\cdot,\cdot]_\theta)$.

The case $r = 2$ is that of an adapted almost Lie structure $(\eta^{(2)}, \Delta^{(2)}, [\cdot,\cdot]_{\eta^{)2)}})$ as described above, with $\xi = \theta$, when $E\xi = \eta^{(2)}$. The adapted almost Lie structure is defined in this case by an R-connection on θ, using the anchored vector bundle $(\theta \otimes \theta, \mathcal{D})$, i.e. an connection in the bundle $\eta^{(2)}_{\theta\otimes\theta}$.

We suppose that the adapted almost Lie structure $(\eta^{(k)}, \Delta^{(k)}, [\cdot,\cdot]_{\eta^{(k)}})$ was constructed for $2 \leq k \leq r-1$, using the anchored vector bundle $(\theta \otimes \theta, \mathcal{D})$, successively, and the connections:

(1) $C^{(2)}$ on $\eta^{(2)}_{\theta\otimes\theta}$, having as components $\left\{ \underset{1}{C}{}^{\alpha}_{\beta\gamma} \right\}$;

$\vdots$

(2) $C^{(r-1)}$ on $\eta^{(r-1)}_{\theta\otimes\theta}$, having as components $\left\{ \underset{1}{C}{}^{\alpha}_{\beta\gamma}, \dots, \underset{r-1}{C}{}^{\alpha}_{\beta\gamma} \right\}$.

The adapted bracket $[\cdot,\cdot]_{\eta^{(r-1)}}$ has the form

$$[\bar{s}_\beta, \bar{s}_\gamma]_{\eta^{(r-1)}} = B^\alpha_{\beta\gamma}\bar{s}_\alpha + \underset{1}{C}{}^{\alpha}_{\beta\gamma} S^{(1)}_\alpha + \dots + \underset{r-1}{C}{}^{\alpha}_{\beta\gamma} S^{(r)}_\alpha,$$

$$[\bar{s}_\alpha, S^{(u)}_\beta]_{\eta^{(r-1)}} = [S^{(u)}_\alpha, S^{(v)}_\beta]_{\eta^{(r-1)}} = 0, \qquad (\forall)\alpha,\beta = \overline{1,m},\ u,v = \overline{1,r-1}.$$

Considering a connection $C^{(r)}$ on $\eta^{(r)}_{\theta\otimes\theta}$, having as components $\left\{ \underset{1}{C}{}^{\alpha}_{\beta\gamma}, \dots, \underset{r}{C}{}^{\alpha}_{\beta\gamma} \right\}$, one construct, as in the case $r-1$, the adapted bracket on $\eta^{(r)}_{\theta\otimes\theta}$.

Notice that the following result is an immediate consequence of the above constructions.

Proposition 3.3. *In the case when $(\theta, D, [\cdot,\cdot]_\theta)$ is an algebroid ($\mathcal{D} = 0$) or a Lie algebroid ($\mathcal{D} = 0$ and $\mathcal{J} = 0$), the components of the connections $C^{(1)}, C^{(2)}, \dots, C^{(r)}$ can be taken zero and the almost Lie structures $(\eta^{(k)}, \Delta^{(k)}, [\cdot,\cdot]_{\eta^{(k)}})$, $1 \le k \le r-1$, are algebroids, respectively Lie algebroids.*

Since the constructions involving r-connections (and also r-Lagrangians and r-sprays, as in [18]) are based in principal on the bundles $\eta^{(r)}$, it is necessary to find some isomorphisms of the vector bundles $\xi^{(r)}$ and $\eta^{(r)}$.

3.4. **Second and third order isomorphisms of the vector bundles $\xi^{(r)}$ and $\eta^{(r)}$, defined by almost Lie structures.** In this subsection we construct isomorphisms of the vector bundles $\xi^{(r)}$ and $\eta^{(r)}$ for $r = 2$ and $r = 3$, which depends only on the adapted almost Lie structure. The general case $r > 3$ is still a mystery for us.

According to the formulas (3.1) and (3.3), the cocycles of the vector bundles $\xi^{(2)}$ and $\eta^{(2)}$ are given by the second order matrices:

$$\overline{\mathcal{M}}_{\bar{U}\cap U} = \begin{pmatrix} g^{\alpha'_0}_{\beta_0} & 0 \\ \dfrac{\partial y^{(2)\alpha'_1}}{\partial y^{(1)\beta_0}} & g^{\alpha'_1}_{\beta_1} \end{pmatrix} \tag{3.5}$$

and

$$A_{\bar{U}\cap U} = \begin{pmatrix} g^{\alpha'_0}_{\beta_0} & 0 \\ D^i_{\beta_0}\dfrac{\partial y^{(1)\alpha'_1}}{\partial x^i} & g^{\alpha'_1}_{\beta_1} \end{pmatrix}. \tag{3.6}$$

respectively. There are two exact sequences of vector bundles over E:

$$0 \to V\theta^{(1)} \to \xi^{(2)} \to p^*\theta \to 0,$$
$$0 \to V\theta^{(1)} \to \eta^{(2)} \to p^*\theta \to 0$$

Proposition 3.4. *Let $(\theta, D, [\cdot,\cdot]_\theta)$ be an almost Lie structure. Then there is an isomorphism of the vector bundles $\xi^{(2)}$ and $\eta^{(2)}$ which depends only on the almost Lie structure on θ.*

Proof. Using the Appendix, it suffices to find a left splitting of the exact sequence $0 \to V\theta^{(2)} \to (\eta^{(2)} - \xi^{(2)}) \to p^*\theta \to 0$, which depends only on the almost Lie structure. The structural functions of $(\eta^{(1)} - \xi^{(1)})$ are

$$\begin{pmatrix} g_{\beta_0}^{\alpha'_0} & 0 \\ D_{\beta_0}^i \dfrac{\partial y^{(1)\alpha'_1}}{\partial x^i} - \dfrac{\partial y^{(2)\alpha'_1}}{\partial y^{(1)\beta_0}} & g_{\beta_1}^{\alpha'_1} \end{pmatrix}.$$

Using local coordinates, the local form of this splitting can be taken $N_\alpha^\beta = L_{\alpha\gamma}^\beta y^{(1)\gamma}$, where $\{L_{\alpha\gamma}^\beta\}$ are the local components of the bracket $[\cdot,\cdot]_\theta$. □

According to the formulas (3.1) and (3.3), the cocycles of the vector bundles $\xi^{(3)}$ and $\eta^{(3)}$ are given by the third order matrices:

$$\overline{\mathcal{M}}_{\bar{U}\cap U} = \begin{pmatrix} \dfrac{\partial y^{(1)\alpha'_0}}{\partial y^{(1)\beta_0}} & 0 & 0 \\ \dfrac{\partial y^{(2)\alpha'_1}}{\partial y^{(1)\beta_0}} & \dfrac{\partial y^{(1)\alpha'_1}}{\partial y^{(1)\beta_1}} & 0 \\ \dfrac{\partial y^{(3)\alpha'_2}}{\partial y^{(1)\beta_0}} & \dfrac{\partial y^{(2)\alpha'_2}}{\partial y^{(1)\beta_1}} & \dfrac{\partial y^{(2)\alpha'_2}}{\partial y^{(2)\beta_2}} \end{pmatrix} \tag{3.7}$$

and

$$A_{\bar{U}\cap U} = \begin{pmatrix} g_{\beta_0}^{\alpha'_0} & 0 & 0 \\ D_{\beta_0}^i \dfrac{\partial y^{(1)\alpha'_1}}{\partial x^i} & \dfrac{\partial y^{(1)\alpha'_1}}{\partial y^{(1)\beta_1}} & 0 \\ D_{\beta_0}^i \dfrac{\partial y^{(2)\alpha'_1}}{\partial x^i} & \dfrac{\partial y^{(2)\alpha'_r}}{\partial y^{(1)\beta_1}} & \dfrac{\partial y^{(2)\alpha'_r}}{\partial y^{(2)\beta_r}} \end{pmatrix} \tag{3.8}$$

respectively. There are two exact sequences of vector bundles over E:

$$0 \to V\theta^{(3)} \to \xi^{(3)} \to p^*\theta \to 0,$$
$$0 \to V\theta^{(3)} \to \eta^{(3)} \to p^*\theta \to 0$$

Proposition 3.5. *Let $(\theta, D, [\cdot,\cdot]_\theta)$ be an almost Lie structure. Then there is an isomorphism of the vector bundles $\xi^{(3)}$ and $\eta^{(3)}$ which depends only on the almost Lie structure on $\eta^{(2)}$.*

Proof. Using the Appendix, it suffices to find a left splitting of the exact sequence $0 \to V\theta^{(3)} \to (\eta^{(3)} - \xi^{(3)}) \to p^*\theta \to 0$, which depends only on the almost Lie structure. Using local coordinates, the local form of this splitting can be taken $N_\alpha^\beta = L_{\alpha\gamma}^\beta y^{(1)\gamma}$, where $\{L_{\alpha\gamma}^\beta\}$ are the local components of the bracket $[\cdot,\cdot]_\theta$.

The structural function of $(\eta^{(3)} - \xi^{(3)})$ are

$$\overline{\mathcal{M}}_{\bar{U}\cap U} = \begin{pmatrix} \dfrac{\partial y^{(1)\alpha'_0}}{\partial y^{(1)\beta_0}} & 0 & 0 \\ D^i_{\beta_0}\dfrac{\partial y^{(1)\alpha'_1}}{\partial x^i} - \dfrac{\partial y^{(2)\alpha'_1}}{\partial y^{(1)\beta_0}} & \dfrac{\partial y^{(1)\alpha'_1}}{\partial y^{(1)\beta_1}} & 0 \\ D^i_{\beta_0}\dfrac{\partial y^{(2)\alpha'_1}}{\partial x^i} - \dfrac{\partial y^{(3)\alpha'_2}}{\partial y^{(1)\beta_0}} & \dfrac{\partial y^{(2)\alpha'_2}}{\partial y^{(1)\beta_1}} & \dfrac{\partial y^{(2)\alpha'_2}}{\partial y^{(2)\beta_2}} \end{pmatrix}.$$

We must find some local functions which verify a relation (4.2) for the above structural function. Consider a linear R-connection on θ related by the anchor $\mathcal{D}$ on $\theta \otimes \theta$, which defines an adapted almost Lie structure on $\eta^{(2)}$; denote the local components of this linear R-connection by $\{B^{\alpha}_{\beta\gamma}\}$ and the local components of the bracket $[\cdot,\cdot]_\theta$ by $L^{\gamma}_{\alpha\beta}$. We define

$$N^{\beta}_{\alpha} = y^{(1)\theta} D^i_{\theta} L^{\beta}_{\alpha\gamma,i} y^{(1)\gamma} + B^{\beta}_{\alpha\theta} y^{(1)\theta} + L^{\beta}_{\alpha\theta} y^{(2)\theta}.$$

By a straightforward and long computation, one verify that these local functions and those that have been found in the previous step, define a left splitting of the exact sequence $0 \to V\theta^{(3)} \to (\eta^{(3)} - \xi^{(3)}) \to p^*\theta \to 0$, thus an isomorphism of the vector bundles $\xi^{(3)}$ and $\eta^{(3)}$, which depends only on an adapted almost Lie structure on $\eta^{(1)}$. □

4. APPENDIX

4.1. **Isomorphisms of vector bundles and exact sequences.**

Lemma 4.1. *Let* $0 \to \xi_1 \to \xi_2 \to \xi_3 \to 0$ *and* $0 \to \eta_1 \to \eta_2 \to \eta_3 \to 0$ *be two exact sequences of vector bundle over the same base. If* ξ_1 *and* ξ_3 *are isomorphic to the vector bundles* η_1 *and* η_3*, respectively, then the vector bundles* ξ_2 *and* η_2 *are isomorphic.*

In fact, the vector bundles ξ_2 and η_2 are isomorphic with the Whitney sums $\xi_1 \oplus \xi_3$ and $\eta_1 \oplus \eta_3$, the isomorphisms being given, for example, by two splittings of the exact sequences. It is interesting that in fact there is sufficient to give only one appropriate splitting. We prove that if $\xi_1 = \eta_1$ and $\xi_3 = \eta_3$, then in order to get an isomorphism of ξ_2 and η_2, it suffices to consider only one splitting, but of an other exact sequence. Since most of constructions performed in the paper use local coordinates, let us consider some coordinates. The local coordinates on the base can be chosen such that the corresponding cocycles of ξ_1, ξ_3, ξ_2 and η_2 are $\{(g^{\alpha'}_{\alpha})\}$, $\{(h^{a'}_{a})\}$, $\left\{\begin{pmatrix} g^{\alpha'}_{\alpha} & 0 \\ u^{a'}_{\alpha} & h^{a'}_{a} \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} g^{\alpha'}_{\alpha} & 0 \\ v^{a'}_{\alpha} & h^{a'}_{a} \end{pmatrix}\right\}$ respectively . Then the change rules of these cocycles, on the intersection of two domains on the base are:

$$\begin{aligned} g^{\alpha''}_{\alpha'} g^{\alpha'}_{\alpha} &= g^{\alpha''}_{\alpha}, \\ h^{a''}_{a'} h^{a'}_{a} &= h^{a''}_{a}, \\ u^{a''}_{a} &= u^{a''}_{\alpha'} g^{\alpha'}_{\alpha} + h^{a''}_{b'} u^{b'}_{\alpha}, \\ v^{a''}_{a} &= v^{a''}_{\alpha'} g^{\alpha'}_{\alpha} + v^{a''}_{b'} u^{b'}_{\alpha}. \end{aligned}$$

Notice that a splitting of the exact sequence $0 \to \xi_1 \to \xi_2 \to \xi_3 \to 0$ is uniquely determined by a set of local functions $\{N^a_\alpha\}$ defined of the domains of the adapted coordinates, which change on the intersection of two domains according to the rule $N^{a'}_{\alpha'} g^{\alpha'}_{\alpha} = h^{a'}_{a} N^a_\alpha - u^{a'}_{a}$.

It is easy to see that $\{w^{a'}_{a} = v^{a'}_{a} - u^{a'}_{a}\}$ defines a cocycle $\left\{\begin{pmatrix} g^{\alpha'}_{\alpha} & 0 \\ w^{a'}_{\alpha} & h^{a'}_{a} \end{pmatrix}\right\}$ of an other vector bundle over the same base, denoted as $\eta_2 - \xi_2$. It follows an exact sequence of vector bundles of the form:

$$0 \to \xi_1 \to (\eta_2 - \xi_2) \to \xi_3 \to 0. \tag{4.1}$$

Lemma 4.2. *A splitting of the exact sequence of vector bundles* (4.1) *defines an isomorphism of the vector bundles* ξ_2 *and* η_2.

Proof. A left splitting of the exact sequence of vector bundles (4.1) has the following local form on the fibres: $(X^\alpha, Y^a) \to (Y^a + N^a_\alpha X^\alpha)$. The change rule of the local functions $\{N^a_\alpha\}$ is

$$N^{a'}_{\alpha'} g^{\alpha'}_{\alpha} = h^{a'}_{a} N^a_\alpha - (v^{a'}_{\alpha} - u^{a'}_{\alpha}). \tag{4.2}$$

Consider on the same domain U of the base, the local matrix given by the formula $\Phi_U = \begin{pmatrix} \delta^\alpha_\beta & 0 \\ N^a_\beta & \delta^a_b \end{pmatrix}$. The relation (4.2) shows that $\begin{pmatrix} g^{\alpha'}_{\beta} & 0 \\ u^{a'}_{\beta} & h^{a'}_{b} \end{pmatrix} \begin{pmatrix} \delta^\beta_\alpha & 0 \\ N^b_\alpha & \delta^b_a \end{pmatrix} = \begin{pmatrix} \delta^{\alpha'}_{\beta'} & 0 \\ N^{a'}_{\beta'} & \delta^{a'}_{b'} \end{pmatrix} \begin{pmatrix} g^{\beta'}_{\alpha} & 0 \\ v^{b'}_{\alpha} & h^{b'}_{a} \end{pmatrix}$, thus the matrices Φ_U define a global isomorphism of the vector bundles ξ_2 and η_2. □

REFERENCES

[1] A. Bejancu: *Vectorial Finsler connections and theory of Finsler subspaces*, Seminar on Geometry and Topology, Timişoara, 1986.

[2] M. B. Boyom: *Anchored vector bundles and algebroids*, arXiv:math.DG/0208012

[3] I. Bucataru: *Horizontal lift in the higher order geometry*, Publ. Math. Debrecen, **56**(2000), No. 1-2, 21-32.

[4] R. L. Fernandes: *Lie algebroids, holonomy and characteristic classes*, Adv. in Math., **70**(2002), 119-179; arXiv:math-DG 0007132.

[5] F. Cantrijn and B. Langerock: *Generalised Connections over a Vector Bundle Map*, Diff. Geom. Appl., **18**(2003), 295-317; arXiv: math.DG/0201274.

[6] M. de León, J. C. Marrero and E. Martínez: *Lagrangian submanifolds and dynamics on Lie algebroids*, J. Phys. A, Math. Gen., **24**, 38 (2005) R241-R308, arXiv:math. DG/0407528, v1 30 Jul 2004.

[7] R. Miron: *The Geometry of Higher Order Lagrange Spaces. Applications to Mechanics and Physics*, Kluwer, Dordrecht, FTPH no. 82, 1997.

[8] R. Miron and Gh. Atanasiu: *Compendium on the higher order Lagrange spaces,* Tensor, N.S., **53**(1993), 39-57.

[9] R. Miron and Gh. Atanasiu: *Differential geometry of the k-osculator bundle*, Rev. Roum. Math. Pures Appl., **41**(1996), No. 3-4, 205-236.

[10] R. Miron and M. Anastasiei: *Vector bundles. Lagrange spaces. Applications to the theory of relativity*, Ed. Academiei, Bucureşti, 1987.

[11] R. Miron and M. Anastasiei: *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publ., 1994.

[12] Marcela Popescu: *Connections on Finsler bundles*, The Second Int. Workshop Diff. Geom. Appl., September 25-28, 1995, Constanţa, An. St. Univ. *Ovidius* Constanţa, s.mat., vol III, fasc. 2 (1995), 97-101.

[13] Marcela Popescu and P. Popescu: *Geometrical objects on anchored vector bundles*, Lie Algebroids and Related Topics in Diff. Geom., J. Kubarski, P. Urbanski and R. Wolak (eds.), Banach Center Publ., **54**(2001), 217-233.

[14] P. Popescu: *On the geometry of relative tangent spaces*, Rev. Roum. Math. Pures Appl., **37**(1992), No. 8, 727-733.

[15] P. Popescu: *Almost Lie structures, derivations and R-curvature on relative tangent spaces*, Rev. Roum. Math. Pures Appl., **37**(1992), No. 8, 779-789.

[16] P. Popescu: *On quasi-connections on fibered manifolds*, New Developements in Diff. Geom., Kluwer Academic Publ., 350, 1996, 343-352.

[17] P. Popescu: *Categories of modules with differentials*, Journal of Algebra, **185**(1996), 50-73.

[18] P. Popescu: *On higher order geometry on anchored vector bundles*, Cent. Eur. J. Math., **5**(2004), No. 2, 826-839.

[19] P. Popescu and Marcela Popescu: *A general background of higher order geometry and induced objects on subspaces*, Balkan J. Geom. Appl., **7**(2002), No. 1, 79-90.

[20] Y.-C. Wong: *Linear connections and quasi connections on a differentiable manifold*, Tôhoku Math J., **14**(1962), 49-63.

University of Craiova
Department of Applied Mathematics
13*, Al. I. Cuza st., Craiova,* 200585*, Romania*
E-mail address: `paul_p_popescu@yahoo.com`

University of Craiova
Department of Applied Mathematics
13*, Al. I. Cuza st., Craiova,* 200585*, Romania*
E-mail address: `marcelacpopescu@yahoo.com`