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ON A CLASS OF GENERAL VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce a new class of variational inequalities, which is called the general variational inequality. We also introduce a new class of Wiener-Hopf equations. Essentially using the projection technique, we establish the equivalence between the general variational inequalities and the fixed point problems as well as with the Wiener-Hopf equations. This equivalent formulation is used to suggest and analyze some iterative algorithms for solving the general variational inequalities. We also discuss the convergence analysis of these iterative methods. Several special cases are also discussed. Our methods of proofs are very simple as compared with other techniques.

1. INTRODUCTION

Variational inequalities, which were introduced and studied by Stampacchia [32] in early sixties, are being used to study a wide class of problems, which arise in various branches of pure and applied sciences. Variational inequalities combine novel theoretical and algorithmic advances with new domain of applications. As a result of interaction among different branches of mathematical and engineering sciences, considerable interest has been shown in developing various extensions and generalizations of variational inequalities, both for their own sake and for their applications. There are significant developments of these problems related to nonconvex optimization, iterative method and structural analysis, see [1-32] and the references therein.

Variational inequalities represent the optimality conditions for a differentiable convex functions on a convex sets in normed space. It is known that the properties of the solutions of the variational inequalities may not hold, in general, when the convex set is nonconvex. In the recent years, the concept of convexity has been generalized in several directions, see, for example, [2] and the references therein. A significant generalization of the convex set is the introduction of the g-convex set [2] and g-convex function [21]. It has been shown [21] that these nonconvex functions enjoy some nice properties which convex function have. We would like to emphasize that the g-convex set and g-convex functions may not be convex sets and convex functions. In this paper, we show that the minimum of a differentiable g-convex

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function on the g-convex set can be characterized by a class of variational inequalities. This fact has motivated us to introduce and consider a new class of variational inequalities, which is called the general nonlinear variational inequalities. In passing, we would like to mention that this new class of general variational inequalities is quite and remarkedably different from the general variational inequalities, which were introduced and studied by Noor [6,7,20]. Related to the general variational inequalities, we also consider a new class of the Wiener-Hopf equations, which is called the general Wiener-Hopf equations. We also discuss its special cases.

Several numerical methods including the projection method and its variant forms, Wiener-Hopf equations, auxiliary principle technique and merit function technique have been developed for solving variational inequalities and related optimization problems. We would like to mention that the Projection method represents an important tool for finding the approximate solution of various types of variational inequalities. The projection type methods were developed in 1970's. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed point problem using the concept of projection. These methods have been extended and modified in various ways. Shi [31] considered the problem of solving a system of nonlinear projections, which are called the Wiener-Hopf equations. It has been shown by Shi [31] that the Wiener-Hopf equations are equivalent to the variational inequalities. It have been shown that the Wiener-Hopf equations provide us a simple, natural, elegant and convenient device to develop some efficient numerical methods for solving variational and complementarity problems, see [6, 8,9 11-28] and the references therein. Essentially using the projection technique, we prove that the general variational inequalities are equivalent to the fixed point problems and the Wiener-Hopf equations. This alternative equivalent formulation is used to discuss the existence of a solution of the general variational inequalities as well as to suggest and analyze some iterative methods for solving general variational inequalities. We also study the convergence analysis of these new iterative methods under suitable conditions. The ideas and techniques of this paper may be starting point for a wide range of novel and innovative applications in various fields.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let K be a nonempty closed convex set in H.

For given nonlinear operators $T, g \colon H \to H$, we consider the problem of finding $u \in H \colon g(u) \in K$ such that

$$\langle \rho Tu, +g(u) - u), v - g(u) \rangle \geq 0, \quad \forall v \in K \tag{2.1}$$

where $\rho > 0$ is a constant. Variational inequality of the type (2.1) is called the general variational inequality involving the two nonlinear operators.

We note that, if $u = g(u)$, then problem (2.1) is equivalent to finding $u \in H \colon g(u) \in K$ such that

$$\langle T(g(u)), v - g(u) \rangle \geq 0, \quad \forall v \in K. \tag{2.2}$$

Inequality of type (2.2) is also called the *general variational inequality involving two operators,* which was introduced and studied by Noor [7] in 1988. For the numerical analysis, applications and other aspects of these variational inequalities, see [7,22,27] and the references therein.

To convey an idea of the applications of the variational inequality (2.2), we show that the minimum of a differentiable nonconvex function on a nonconvex set K in H can be characterized by the general variational inequality (2.2) and this is the main motivation of our next result (Lemma 2.1).

For this purpose, we recall the following well known concepts, see [2].

Definition 2.1. Let K be any set in H. The set K is said to be g-convex, if there exist a function $g\colon H \to H$ such that

$$g(u) + t(v - g(u)) \in K, \quad \forall u, v \in H\colon g(u),\ v \in K,\ t \in [0,1].$$

Note that every convex set is g-convex, but the converse is not true, see[2]. If $g = I$, then the g-convex set K is called the convex set.

Definition 2.2. The function $F\colon K \to H$ is said to be g-convex, if there exists a function g such that

$$F(g(u) + t(v - g(u))) \le (1-t)F(g(u)) + tF(v), \quad \forall u, v \in H\colon g(u), v \in K,\ t \in [0,1].$$

Clearly every convex function is g-convex, but the converse is not true. For the properties and various classes of the g-convex functions, see Noor [21].

We now state and main result of this section and this is the motivation of our next result.

Lemma 2.1. *Let $F\colon K \to H$ be a differentiable g-convex function. Then $u \in H$: $g(u) \in K$ is the minimum of g-convex function F on $K \subset g(H)$, if and only if $u \in H\colon g(u) \in$satisfies the inequality*

$$\langle F'(g(u)), v - g(u)\rangle \ge 0, \quad \forall v \in K, \tag{2.3}$$

where $F'(g(u))$ is the differential of F at $g(u) \in K$ in the direction of $v - g(u)$.

Proof. Let $u \in H\colon g(u) \in K$ be a minimum of g-convex function F on K. Then

$$F(g(u)) \le F(v), \quad \forall v \in K. \tag{2.4}$$

Since K is a g-convex set, so, for all $u, v \in H\colon g(u),\ v \in K,\ t \in [0,1],\ g(v_t) = g(u) + t(v - g(u)) \in K$. Setting $g(v) = g(v_t)$ in (2.4), we have

$$F(g(u)) \le F(g(u) + t(v - g(u))).$$

Dividing the above inequality by t and taking $t \to 0$, we have

$$\langle F'(g(u)), v - g(u)\rangle \ge 0, \quad \forall v \in K,$$

which is the required result (2.3).

Conversely, let $u \in H\colon g(u) \in K$ satisfy the inequality (2.3). Since F is a g-convex function, $\forall u, v \in H\colon v,\ g(u) \in K$, $t \in [0,1]$, $g(u) + t(v - g(u)) \in K$ and

$$F(g(u) + t(v - g(u))) \le (1-t)F(g(u)) + tF(v),$$

which implies that

$$F(v) - F(g(u)) \geq \frac{F(g(u) + t(v - g(u)) - F(g(u))}{t}.$$

Letting $t \to 0$, we have

$$F(v) - F(g(u)) \geq \langle F'(g(u)), v - g(u) \rangle \geq 0, \quad \text{using (2.3)},$$

which implies that

$$F(g(u)) \leq F(v), \quad \forall v \in K$$

showing that $u \in K$ is the minimum of F on K in H. □

Lemma 2.1 implies that g-convex programming problem can be studied via the general variational inequality (2.2) with $T(g(u)) = F'(g(u))$. In a similar way, one can show that the general variational inequality is the Fritz-John condition of the inequality constrained optimization problem.

For $g = I$, the identity operator, the general variational inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K, \tag{2.5}$$

which is known as the classical variational inequality and was introduced in 1964 by Stampacchia [32]. For the recent applications, numerical methods, sensitivity analysis, dynamical systems and formulation of variational inequalities, see [1, 3-31] and the references therein.

We also need the following concepts and results.

Lemma 2.2. *Let K be a closed convex set in H. Then, for a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,$$

if and only if

$$u = P_K z,$$

where P_K is the projection of H onto the closed convex set K in H. It is well known that the projection operator P_K is a nonexpansive operator.

Related to the general variational inequalities (2.1), we consider the problem of solving the Wiener-Hopf equations. To be more precise, let $Q_K = I - g^{-1}P_K$, where I is the identity operator and g is a given nonlinear operator such that its inverse exists. For given nonlinear operators T, g, we consider the problem of finding $z \in H$ such that

$$Tg^{-1}P_K z + \rho^{-1}Q_K z = 0, \tag{2.6}$$

which is called the general Wiener-Hopf equation. We note that, if $g = I$, then one can obtain the original Wiener-Hopf equations, which are mainly due to Shi [31]. It has been shown that the Wiener-Hopf equations have played an important and significant role in developing several numerical techniques for solving variational inequalities and related optimization problems, see [8-28,30,31].

Definition 2.3. For all $u, v \in H$, an operator $T\colon H \to H$ is said to be:
(i) *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha||u - v||^2$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$||Tu - Tv|| \leq \beta||u - v||.$$

From (i) and (ii), it follows that $\alpha \leq \beta$.

3. PROJECTION OPERATOR METHOD

In this section, we suggest and analyze some new approximation schemes for solving the general variational inequality (2.1). One can prove that the general variational inequality (2.1) is equivalent to the fixed point problem by invoking Lemma 2.2.

Lemma 3.1. *The function $u \in H\colon g(u) \in K$ is a solution of the special general variational inequality* (2.1) *if and only if $u \in H$ satisfies the relation*

$$g(u) = P_K[u - \rho Tu], \tag{3.1}$$

where P_K is the projection operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the special general variational inequality (2.1) is equivalent to the fixed point problem (3.1). This alternative equivalent formulation is very useful from the numerical and theoretical points of view.

We rewrite the relation (3.1) in the following form

$$F(u) \equiv u = u - g(u) + P_K[u - \rho Tu], \tag{3.2}$$

which is used to study the existence of a solution of the general mixed variational inequality (2.1) and this is the main motivation of our next result.

Theorem 3.1. *Let the operators $T, g\colon H \to H$ be both strongly monotone with constants $\alpha > 0$, $\sigma > 0$ and Lipschitz continuous with constants with $\beta > 0$, $\delta > 0$ respectively. If*

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2-k)}, \quad k = \sqrt{1 - 2\sigma + \delta^2}, < 1, \tag{3.3}$$

$0 \leq \alpha_n \leq 1$, *for all $n \geq 0$ and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *then there exists a solution of problem* (2.1).

Proof. From Lemma 3.1, it follows that problems (2.1) and (3.1) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (3.2) has a fixed point. For all $u \neq v \in H$, we have

$$\begin{aligned} \|F(u) - F(v)\| &= |u - v - (g(u) - g(v))| + |P_K[u - \rho Tu] - P_K[v - \rho Tv]| \\ &\leq \|u - v - (g(u) - g(v))\| + \|u - v - \rho(Tu - Tv)\|, \end{aligned} \tag{3.4}$$

where we have used the fact that the projection operator P_K is nonexpansive.

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned} \|u - v - \rho(Tu - Tv)\|^2 &\leq \|u - v\|^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2\|Tu - Tv\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u - v\|^2. \end{aligned} \tag{3.5}$$

In a similar way, using the strongly monotonicity T with constant $\sigma > 0$ and Lipschitz continuity of T with constant $\delta > 0$, we have

$$\|u - v - (g(u) - g(v))\|^2 \leq (1 - 2\sigma + \delta^2)\|u - v\|^2, \tag{3.6}$$

From (3.4), (3.5) and (3.6), we have

$$\begin{aligned} \|F(u) - F(v)\| &\leq \left\{\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\right\}\|u - v\| \\ &= (k + t(\rho))\|u - v\| = \theta\|u - v\|, \end{aligned}$$

where

$$t(\rho) = \sqrt{1 - 2\alpha\rho + \rho^2\beta^2.} \tag{3.7}$$

and

$$\theta = k + t(\rho). \tag{3.8}$$

From (3.3), it follows that $\theta < 1$. Thus the mapping $F(u)$, defined by (3.2) is a contraction mapping and consequently has a fixed point belonging to H satisfying the general mixed variational inequality (2.1). □

Using the fixed point formulation (3.1), we suggest and analyze the following iterative method for solving the general variational inequalities (2.1).

Algorithm 3.1. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n \left\{u_n - g(u_n) + P_K[u_n - \rho Tu_n]\right\}, \quad n = 0, 1, \ldots \tag{3.9}$$

which is known as the Mann iteration process for solving th general variational inequalities (2.1).

Note that if $g = I$, then Algorithm 3.1 reduces to the following iterative method for solving the variational inequalities (2.5).

Algorithm 3.2. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n P_K[u_n - \rho Tu_n], \quad n = 0, 1, \ldots$$

We now consider the convergence analysis of Algorithm 3.1 and this is the main motivation of our next result.

Theorem 3.2. *Let the operators T, g satisfy all the assumptions of Theorem* 3.1. *If the condition* (3.3) *holds and $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution u_n obtained from Algorithm* 3.1 *converge to a solution $u \in H$ satisfying the general mixed variational inequality* (2.4).

Proof. Let $u \in H\colon g(u) \in K$ be a solution of the general variational inequality (2.4). Then, using Lemma 3.1, we have

$$u = (1-\alpha_n)u + \alpha_n \left\{u - g(u) + P_K[u - \rho Tu]\right\}, \tag{3.10}$$

where $0 \leq \alpha_n \leq 1$ is a constant.

From (3.9) and (3.10), we have

$$\begin{aligned} \|u_{n+1} - u\| &= \big\|(1-\alpha_n)(u_n - u) + \alpha_n \|u_n - u - (g(u_n) - g(u)\| \\ &\qquad + \alpha_n\{P_K[u_n - \rho Tu_n] - P_K[u - \rho Tu]\}\big\| \\ &\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\|u_n - u - (g(u_n) - g(u))\| \\ &\qquad + \alpha_n\|u_n - u - \rho(Tu_n - Tu)\|. \end{aligned} \tag{3.11}$$

From (3.5),(3.6), (3.7), (3.8) and (3.11), we have

$$\begin{aligned} \|u_{n+1} - u\| &\leq (1-\alpha_n)\|u_n - u\| + \left\{\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\right\}\|u_n - u\| \\ &= (1-\alpha_n)\|u_n - u\| + (k + t(\rho))\|u_n - u\|, \\ &= (1-\alpha_n)\|u_n - u\| + \theta\|u_n - u\|. \end{aligned}$$

From (3.3), it follows that $\theta < 1$. Thus

$$\begin{aligned} \|u_{n+1} - u\| &\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\theta\|u_n - u\| = [1 - (1-\theta)\alpha_n]\|u_n - u\| \\ &\leq \prod_{i=0}^{n}[1 - (1-\theta)\alpha_i]\|u_0 - u\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta > 0$, we have $\lim_{n\to\infty} \left\{\prod_{i=0}^{n}[1 - (1-\theta)\alpha_i]\right\} = 0$. Consequently the sequence $\{u_n\}$ convergences strongly to $u \in H$ satisfying the general variational inequality (2.1). This completes the proof. □

4. WIENER-HOPF EQUATION TECHNIQUE

In this Section, we use the Wiener-Hopf equation technique to suggest and analyze an iterative method for solving the general variational inequality (2.1). For this purpose, we need the following result.

Lemma 4.1. *The solution $u \in H\colon g(u) \in K$ satisfies the general variational inequality* (2.1) *if and only if $z \in H$ is a solution of the general Wiener-Hopf equation* (2.6)*, where*

$$g(u) = P_K z \tag{4.1}$$

$$z = u - \rho T u, \quad \rho > 0, \quad \textit{a constant.} \tag{4.2}$$

Proof. Let $u \in H\colon g(u) \in K$ be a solution of (2.1). Then, from Lemma 3.1, we have

$$g(u) = P_K[u - \rho T u]. \tag{4.3}$$

Let

$$z = u - \rho T u. \tag{4.4}$$

Then

$$g(u) = P_K z. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we have

$$z = u - \rho T u = g^{-1} P_K z - \rho T g^{-1} P_K z,$$

from which it follows that $z \in H$ is a solution of the extended general Wiener-Hopf equation (2.6), the required result. □

Lemma 4.1 implies that the general variational inequalities (2.1) and the Wiener-Hopf equation (2.6) are equivalent. We use this equivalent formulation to suggest a number of iterative methods for solving the general variational inequalities.

I. Using (4.1), the Wiener-Hopf equation(2.6) can be rewritten in the form as:

$$Q_K z = -\rho T g^{-1} P_K z,$$

which implies that

$$z = g^{-1} P_K z - \rho T g^{-1} P_K z = u - \rho T u,$$

This fixed point formulation enables to suggest the following iterative method for solving problem (2.1).

Algorithm 4.1. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$g(u_n) = P_K z_n, \tag{4.6}$$

$$z_{n+1} = (1 - \alpha_n) z_n + \alpha_n \{u_n - \rho T u_n\}, \quad n = 0, 1, \ldots, \tag{4.7}$$

where $0 \le \alpha_n \le 1$, for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

II. By an appropriate and suitable rearrangement of the terms and using (4.1), the Wiener-Hopf equations (2.6) can be written as:

$$z = g^{-1}P_K z - \rho T g^{-1}P_K z + (1-\rho^{-1})Q_K zu - \rho Tu + (1-\rho^{-1})Q_K z,$$

which is another fixed point formulation. Using this fixed point formulation, we can suggest the following iterative method.

Algorithm 4.2. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$\begin{aligned} g(u_n) &= P_K z_n \\ z_{n+1} &= u_n - \rho T u_n + (1-\rho^{-1})Q_K z_n, \quad n = 0, 1, \ldots . \end{aligned}$$

III. If T is linear and T^{-1} exists, then the Wiener-Hopf equation (2.6) can be written as:

$$z = \left(I - \rho T^{-1}\right) Q_K z.$$

This fixed point formulation allows us to suggest the following iterative method for solving the general variational inequalities (2.1).

Algorithm 4.3. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$z_{n+1} = \left(I - \rho T^{-1}\right) Q_K z_n, \quad n = 0, 1, \ldots .$$

For $g = I$, the identity operator, Algorithm 4.1 - Algorithm 4.3 are due to Noor [8-14]. In brief, by an appropriate and suitable rearrangements of the terms of the general Wiener-Hopf equations (2.6), one can suggest and analyze a number of iterative methods for solving the general variational inequalities (2.1) and related optimization problems. For the investigation of such type of projection iterative methods and the verification of their numerical efficiency, further research efforts are needed.

We now consider the convergence analysis of Algorithm 4.1. In a similar way, one can study the convergence analysis of Algorithm 4.2 and Algorithm 4.3.

Theorem 4.1. *Let the operators T, g satisfy all the assumptions of Theorem* 3.1. *If the condition* (3.3) *holds and $0 \le \alpha_n \le 1$, for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution $\{z_n\}$ obtained from Algorithm* 4.1 *converges to a solution $z \in H$ satisfying the Wiener-Hopf equation* (2.6) *strongly.*

Proof. Let $u \in H$ be a solution of (2.4). Then, using Lemma 4.1, we have

$$z = (1-\alpha_n)z + \alpha_n\{u - \rho T u\}, \tag{4.8}$$

where $0 \le \alpha_n \le 1$, and $\sum_{n=0}^{\infty} a_n = \infty$.

From (4.7), (4.8), (3.7) and (3.8), we have

$$\begin{aligned}\|z_{n+1}-z\| &\leq (1-\alpha_n)\|z_n-z\|+\alpha_n\|u_n-u-\rho(Tu_n-Tu)\| \\ &\leq (1-\alpha_n)\|z_n-z\|+\alpha_n\left\{\sqrt{1-2\rho\alpha+\beta^2}\right\}\|u_n-u\|, \qquad (4.9)\end{aligned}$$

Also from (4.5), (4.1) and the nonexpansivity of the projection operator P_K, we have

$$\|u_n-u\| = \|u_n-u-(g(u_n)-g(u))\|+\|P_Kz_n-P_Kz\| \leq k\|u_n-u\|+\|z_n-z\|,$$

which implies that

$$\|u_n-u\| \leq \frac{1}{1-k}\|z_n-z\|. \qquad (4.10)$$

Combining (4.9), and (4.10), we have

$$\|z_{n+1}-z\| \leq (1-\alpha_n)\|z_n-z\|+\alpha_n\theta_1\|z_n-z\|, \qquad (4.11)$$

where

$$\theta_1 = \frac{\sqrt{1-2\rho\alpha+\rho^2\beta^2}}{1-k}.$$

Using (3.3), we see that $\theta_1 < 1$ and consequently

$$\begin{aligned}\|z_{n+1}-z\| &\leq (1-\alpha_n)\|z_n-z\|+\alpha_n\theta_1\|z_n-z\| = [1-(1-\theta_1)\alpha_n]\|z_n-z\| \\ &\leq \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|z_0-z\|.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta > 0$, we have $\lim_{n\to\infty}\left\{\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\right\} = 0$. Consequently the sequence $\{z_n\}$ convergences strongly to $z \in H$, the required result. □

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